Analysis of Dynamic Traffic Demand on a Paradoxical Phenomenon and Reachability With Input Constraints

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INTRODUCTION

Traffic problems, congestion for example, have been widely researched. Drivers behave selfishly so that they choose paths to minimize their own travel time to the destinations. When traffic demands of origin-destinations per unit time are constant, there is a path choice equilibrium where no drivers can decrease their own travel time to the destinations by changing their path choices. This equilibrium is called the user equilibrium (UE) [1]. The stability of UE is shown in [2] under some conditions. However, at UE, the traffic network is not used efficiently, and there can be paradoxical phenomenon called Braess paradox [3] in which removing a road decreases the total travel time of drivers.

To resolve the inefficient use of traffic network, some taxation approaches have been proposed to control the traffic distribution over paths. Notable examples are [4–7]. Among such taxation approaches, there is another approach which controls the demands, instead of the traffic distribution in a constant demand. Reference [8] proposes a control method of demands and explains the benefits of demand control. In addition, an interesting case is shown in [9] in which an increase of traffic on a road can decrease the drivers' total travel time. Reference [10] studies the relation for a simple network between demand and Braess paradox and they say Braess paradox disappears when the demand is large enough. However, the demands do not increase automatically because human do not want to use the network when it is congested. When the dynamic demands are considered, how will the properties of converging points be from the viewpoints of drivers' travel time and the existence of inefficient use of network? Dynamic demands are closely related to departure time choices of drivers. Reference [11, 12] study the stability of departure time choices on a road, but they do not address the case shown in [9]. Monetary incentives are oftern considered for demands control [8]. Hence, to increase demands, the road managers may have to give drivers positive monetary incentives (eg. discount of tolls), however, there can be their budget constraints. In addition, controlling not all origin-destination pairs' demands may be beneficial, for exmaple, from the viewpoint of cost to implement the control.

RESEARCH OBJECTIVES

In this research, using a dynamic traffic demand model which is refered to as the selfish human's decision model, we provide analysis of converging points and their properties in a similar phenomenon to the one shown in [9]. The properties are stability, price of anarchy (PoA). PoA is a measure of how efficiently the traffic network is used [5, 13]. In some cases we should control the demands to converge to an efficient ones, and from the motivations of the input constraints such as budeget constraints, we analyze reachability of the efficient demands with input constraints.

TRAFFIC NETWORK

We model the traffic routing as non-atomic game, where the players are regarded as infinitely small. In this research we consider the directed network shown in Fig. 1. We call the pair of origin-c and destination-c as a commodity $c \in \{1, 2\}$, and the non-negative traffic demand per unit time from origin-c to destination-c is denoted by d_c. Hence, in this network, multiple drivers are going from their origins to their destinations. The flow amount on an edge $e \in E$ is denoted by f_e

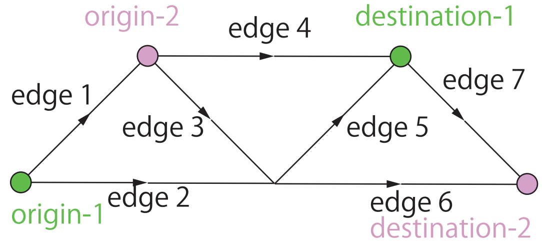


Figure 1: Network graph

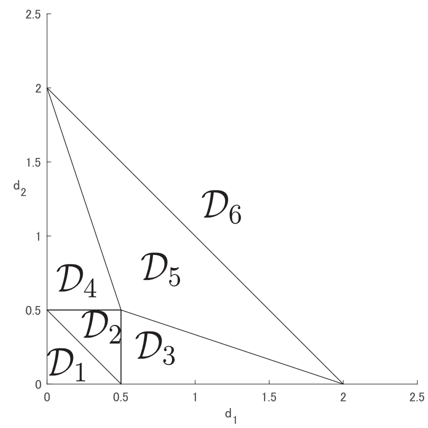


Figure 2: Visualization of regions

where $E =: \{1, \dots, 7\}$ denotes the edge set. The function $l_e(f_e)$ denotes the delay function on edge e which maps f_e to the travel time l_e to pass the edge. Delay functions are defined as

$$l_e(f_e) = \begin{cases} f_e, & e = 1, 3, 5, 7, \quad (1) \\ 1, & e = 2, 4, 6. \quad (2) \end{cases}$$

In this research, we assume that When (d_1, d_2) is given, path flow amount distribution over paths is always at UE.

Expressions of average delay and PoA

$\mathrm{AD}_c(d_1, d_2)$ is the average delay (average travel time from the origin to the desination) of commodity c. We can derive $\mathrm{AD}_c(d_1, d_2)$ at UE as function of d_1, d_2 by solving a convex optimization problem [1]. The expressions of $\mathrm{AD}_1(d_1, d_2), \mathrm{AD}_2(d_1, d_2)$ are different over following 6 regions.

The expressions of $\mathrm{AD}_1(d_1, d_2), \mathrm{AD}_2(d_1, d_2)$ on $\mathcal{D}_5$ is

$$\mathcal{D}_5 : \begin{cases} \mathrm{AD}_1(d_1, d_2) = 1.5 - \dfrac{d_2 - d_1}{4}, & (3) \\ \mathrm{AD}_2(d_1, d_2) = 1.5 - \dfrac{d_1 - d_2}{4}. & (4) \end{cases}$$

Interestingly, for example, on $d_2 = d_1 (\frac{1}{2} \leq d_1 \leq 1)$, the increase of d_1, d_2 do not change the average delays.

Next, we analyze $\mathrm{PoA}(d_1, d_2)$. Total average delay $\mathrm{TAD}(d_1, d_2)$ denotes average delay of all drivers including both commodities at UE. $\mathrm{TAD}^{\min}(d_1, d_2)$ denotes TAD at minimum delay flow (a path flow distribution which minimizes $\mathrm{TAD}(d_1, d_2)$ when (d_1, d_2) is fixed.) $\mathrm{PoA}(d_1, d_2)$ is defined as

$$\mathrm{PoA}(d_1, d_2) := \frac{\mathrm{TAD}\,(d_1, d_2)}{\mathrm{TAD}^{\min}\,(d_1, d_2)} (\geq 1). \tag{5}$$

Network use is more efficient when PoA is smaller. Fig. 3 shows PoA plot. On , PoA is strictly monotonically decreasing. At $(d_1, d_2) = (1, 1)$, PoA is 1.

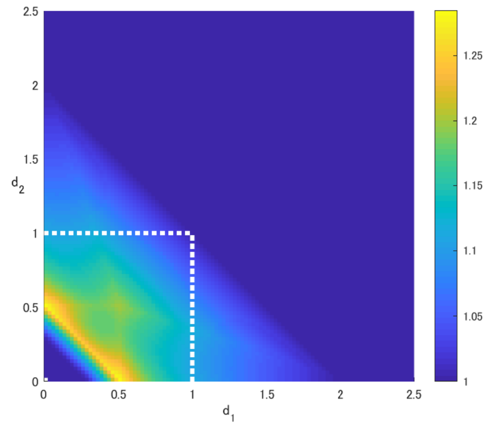
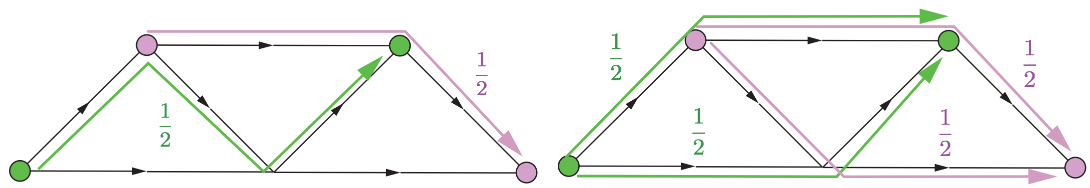


Figure 3: PoA plot



(a) $(d_1, d_2) = (\frac{1}{2}, \frac{1}{2})$

(b) $(d_1, d_2) = (1, 1)$

Figure 4: Path flow amount distributions at UE. $\mathrm{AD}_1 = \mathrm{AD}_2 = 1.5$ in both cases

Fig. 4 shows path flow amount distributions at UE when $(d_1, d_2) = (\frac{1}{2}, \frac{1}{2})$ and $(d_1, d_2) = (1, 1)$. Fig. 4(a) shows an example of the distribution when $(d_1, d_2) = (\frac{1}{2}, \frac{1}{2})$.

Stability and converging points of dynamic demands

Next, we consider following dynamic traffic demand model.

$$\dot{d_1}(t) = s_1 \{1.5 - \mathrm{AD}_1(d(t))\}, \tag{6}$$

$$\dot{d_2}(t) = s_2 \{1.5 - \mathrm{AD}_2(d(t))\}. \tag{7}$$

where s_1, s_2 are positive constants representing the changing speed of each demand. The dynamics represent the human selfish departure time choice. That is, when the average delays are large, less drivers want to make trips than the case when the average delays are small. We proved that on this dynamics, the set of equilibrium points (equilibrium set, ES) is given by $d_2 = d_1(\frac{1}{2} \leq d_1 \leq 1)$. Furthermore, every point in ES is Lyapunov stable. I derived all converging points for all $(d_1, d_2), s_1, s_2$. Then we proved that for any $(d_1, d_2), s_1, s_2$, the trajectory converges to a point on the ES. Fig. 5 shows exmaple trajectories to converging points.

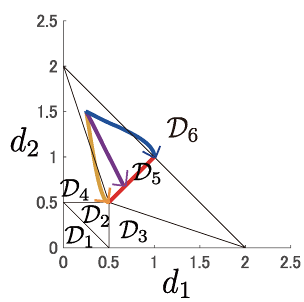


Figure 5: Example trajectories from $(d_1, d_2) = (0.25, 1.5)$. Each trajectory is with different (s_1, s_2). Red line is ES.

Reachability analysis with controll input constraints

On the ES, $\mathrm{AD}_1(d_1, d_2)$ and $\mathrm{AD}_2(d_1, d_2)$ do not change and at $(d_1, d_2) = (1, 1)$, the most drivers can use the network. From the motivation we mention in introduction, we analyze reachability to $(d_1, d_2) = (1, 1)$ with input constraints. I proved that if we control only one of the two demands, $(d_1, d_2) = (1, 1)$ is reachable with any $(d_1, d_2), s_1, s_2$.

Next, we consider zero-sum input controller described as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} s_1\left\{1.5 - \mathrm{AD}_1\left(d_1(t), d_2(t)\right)\right\} \\ s_2\left\{1.5 - \mathrm{AD}_2\left(d_1(t), d_2(t)\right)\right\} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t), \quad u(t) \in \mathbb{R}. \tag{8}$$

$u(t)$ denotes monetary value. I proved that if $s_1 \neq s_2$, (1,1) is reachable from any $(d_1(0), d_2(0))$. If $s_1 = s_2$, (1,1) is reachable from $(d_1(0), d_2(0))\,\{(d_1, d_2)|d_1 + d_2 \leq 2\}$. However, if $s_1 = s_2$ and $d_1(0) + d_2(0) \geq 2$, the trajectory converges to $(d_1(0), d_2(0)) = (1, 1)$ in infinite time. Interestingly, if $s_1 = s_2$ and $d_1(0) + d_2(0) < 2$, the trajectory has to pass (2,0) or (0,2) to reach (1,1).

Conclusion

We analyzed the converging points and stability of equilibrium points of dynamic traffic demands model which is refered to as human selfish departure time choice behavior. Our results show that there is the case in which the demands converge to efficient demands, but in some case not. It is because the equilibrium point is not unique on our dynamics, and it is caused by paradoxical phenomenon in which average delays do not change although the demands increase. Such phenomena is caused by human's selfish path choice. Finally we analyzed the reachability to the efficient demands with simple control input constraints. When we use the zero-sum controller and if $s_1 = s_2$, in some cases the trajectory has to go to $(d_1, d_2) = (2, 0)$ or $(0, 2)$ to reach $(d_1, d_2) = (1, 1)$. Hence at this time the demand of a commodity must be 0. If the drivers of the commodity are unhappy, then we should control the trajectory considering the happiness of road managers and drivers. This can be our next research topic. For other future works, we will extend the network to more complicated one, and the delay functions each road to nonlinear ones. In addition, when we control the demands, considering scale of monetary incentives and convergence speed to the length of the time period is another direction of our research.

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