Neural Network Adaptive Control for a Class of Nonlinear Uncertain Dynamical Systems With Asymptotic Stability Guarantees

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***Abstract*—In this paper, a neuroadaptive control framework for continuous- and discrete-time nonlinear uncertain dynamical systems with input-to-state stable internal dynamics is developed. The proposed framework is Lyapunov based and unlike standard neural network (NN) controllers guaranteeing ultimate boundedness, the framework guarantees partial asymptotic stability of the closed-loop system, that is, asymptotic stability with respect to part of the closed-loop system states associated with the system plant states. The neuroadaptive controllers are constructed without requiring explicit knowledge of the system dynamics other than the assumption that the plant dynamics are continuously differentiable and that the approximation error of uncertain system nonlinearities lie in a small gain-type norm bounded conic sector. This allows us to merge robust control synthesis tools with NN adaptive control tools to guarantee system stability. Finally, two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.**

***Index Terms*—Adaptive control, asymptotic stability, input-to-state stable internal dynamics, neural networks (NNs), partial stability, sector-bounded nonlinearities.**

I. Introduction

ONE of the main motivation for developing neural network (NN) adaptive control algorithms is their capability to approximate a large class of continuous nonlinear maps from the collective action of very simple, autonomous processing units that are connected in simple ways. These processing units involve a weighted interconnection of fundamental elements called *neurons*, which are functions consisting of a summing junction and a nonlinear operation involving an activation function. In addition, NNs have attracted attention due to their inherently parallel and highly redundant processing architecture that makes it possible to develop parallel weight update laws. This parallelism makes it possible to effectively update an NN online. Consequently, the use of the NNs for system identification and control of complex highly uncertain dynamical systems has become an active area of research [1]–[9].

Unlike adaptive controllers which guarantee asymptotic stability of the closed-loop system states associated with the system plant states, standard NN adaptive controllers guarantee *ultimate boundedness* of the closed-loop system states [10]. This fundamental difference between adaptive control and neuroadaptive control can be traced back to the modeling and treatment of the system uncertainties. In particular, adaptive control is based on *constant, linearly parameterized* system uncertainty models of a known structure but unknown variation [11]–[13], while neuroadaptive control is based on the universal function approximation property, wherein any continuous system uncertainty can be *approximated* arbitrarily closely on a compact set using an NN with appropriate weights [14]. This system uncertainty parametrization makes it impossible to construct a system Lyapunov function whose time derivative along the closed-loop system trajectories is guaranteed to be negative definite. Instead, the Lyapunov derivative can only be shown to be negative on a sublevel set of the system Lyapunov function. This shows that, in this sublevel set, the Lyapunov function will decrease monotonically until the system trajectories enter a compact positively invariant set containing the desired system equilibrium point, and thus, guaranteeing ultimate boundedness. This analysis is often conservative since standard Lyapunov-like theorems used to show ultimate boundedness of the closed-loop system states provide only sufficient conditions, while NN controllers often achieve plant state convergence to a desired equilibrium point.

In this paper, we develop a neuroadaptive control framework for a class of nonlinear uncertain dynamical systems which guarantees asymptotic stability of the closed-loop system states associated with the system plant states, as well as boundedness of the NN weighting gains. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system, that is, Lyapunov stability of the overall closed-loop system states and convergence of the plant states [15]. The neuroadaptive controllers are constructed without requiring explicit knowledge of the system dynamics other than the assumption that the plant dynamics are continuously differentiable and that the approximation error of uncertain system nonlinearities lie in a small gain-type norm bounded conic sector. Furthermore, the proposed neurocontrol architecture is modular in the sense that if a nominal linear design model is

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available, then the neuroadaptive controller can be augmented to the nominal design to account for system nonlinearities and system uncertainty. Hence, the approach of the present paper is related to the neuroadaptive control methods developed in [6], [16]. Finally, we emphasize that since our focus in this paper is to develop a neuroadaptive control framework that guarantees asymptotic stability, we limit our attention to NN controllers with full-state feedback and matched uncertainties. Extensions to output feedback neuroadaptive control operating over a tapped delay line of available input and output measurements can be addressed using the techniques in [6] and [17].

The notation used in this paper is fairly standard. Specifically, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{\dagger}$ denotes the Moore–Penrose generalized inverse, $\mathrm{tr}(\cdot)$ denotes the trace operator, $\ln(\cdot)$ denotes the natural log operator, $\sigma_{\max}(\cdot)$ denotes the maximum singular value, $\mathrm{vec}(\cdot)$ denotes the column stacking operator, $\|\cdot\|$ denotes the Euclidean vector norm, and $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) denotes the maximum (respectively, minimum) eigenvalue of a Hermitian matrix.

II. Stable Neuroadaptive Control for Nonlinear Uncertain Systems

In this section, we consider the problem of characterizing neural adaptive feedback control laws for a class of nonlinear uncertain dynamical systems. Specifically, consider the controlled nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f_x(x(t), z(t)) + G(x(t), z(t))u(t), \quad x(0) = x_0, \quad t \geq 0 \quad (1)$$

$$\dot{z}(t) = f_z(x(t), z(t)), \quad z(0) = z_0 \quad (2)$$

where $x(t) \in \mathbb{R}^{n_x}$, $t \geq 0$, and $z(t) \in \mathbb{R}^{n_z}$, $t \geq 0$, are the state vectors, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f_x : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ satisfies $f_x(0, z) = 0$, $z \in \mathbb{R}^{n_z}$, $f_x(\cdot, z)$ is continuously differentiable on $\mathbb{R}^{n_x}$ for each $z \in \mathbb{R}^{n_z}$, $f_x(x, \cdot)$ is continuous on $\mathbb{R}^{n_z}$ for each $x \in \mathbb{R}^{n_x}$, $f_z : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$ satisfies $f_z(x, 0) = 0$, $x \in \mathbb{R}^{n_x}$, $f_z(x, \cdot)$ is Lipschitz continuous in z for each $x \in \mathbb{R}^{n_x}$, and $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x \times m}$ is continuous. The dynamics (2) typically describe the internal dynamics of the system $\mathcal{G}$. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. Furthermore, for the nonlinear uncertain system $\mathcal{G}$, we assume that the required properties for the existence and uniqueness of solutions are satisfied, that is, $f_x(\cdot, \cdot)$, $f_z(\cdot, \cdot)$, $G(\cdot, \cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that (1) and (2) have a unique solution forward in time.

In this paper, we assume that $f_x(\cdot, \cdot)$ and $f_z(\cdot, \cdot)$ are unknown functions, and $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by

$$f_x(x, z) = Ax + \Delta f(x, z) \quad (3)$$

$$G(x, z) = BG_{\mathrm{n}}(x, z) \quad (4)$$

where $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times m}$ are known matrices, $G_{\mathrm{n}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m \times m}$ is a known matrix function such that $\det G_{\mathrm{n}}(x, z) \neq 0$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, and $\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is an uncertain function belonging to the uncertainty set $\mathcal{F}$ given by

$$\mathcal{F} = \{\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x} : \Delta f(0, \cdot) = 0, \quad \Delta f(x, z) = B\delta(x, z), (x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}\} \quad (5)$$

where $\delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^m$ is an uncertain function such that $\delta(\cdot, z)$ is continuously differentiable on $\mathbb{R}^{n_x}$ for each $z \in \mathbb{R}^{n_z}$, $\delta(x, \cdot)$ is continuous on $\mathbb{R}^{n_z}$ for each $x \in \mathbb{R}^{n_x}$, and $\delta(0, \cdot) = 0$. Furthermore, we assume that (2) is input-to-state stable with $x(t)$ viewed as the input. It is important to note that since $\delta(x, z)$ is continuously differentiable in x and $\delta(0, z) = 0$, $z \in \mathbb{R}^{n_z}$, it follows that there exists a continuous matrix function $\Delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m \times n_x}$ such that $\delta(x, z) = \Delta(x, z)x$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. We assume that the continuous matrix function $\Delta(\cdot, \cdot)$ can be approximated over a compact set $\mathcal{D}_{\mathrm{c}_x} \times \mathcal{D}_{\mathrm{c}_z} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ by a linear in the parameters NN up to a desired accuracy so that

$$\mathrm{col}_i(\Delta(x, z)) = W_i^{\mathrm{T}}\sigma(x, z) + \varepsilon_i(x, z), \quad (x, z) \in \mathcal{D}_{\mathrm{c}x} \times \mathcal{D}_{\mathrm{c}z}, \quad i = 1, \ldots, n_x \quad (6)$$

where $\mathrm{col}_i(\Delta(\cdot, \cdot))$ denotes the ith column of $\Delta(\cdot, \cdot)$, $W_i^{\mathrm{T}} \in \mathbb{R}^{m \times s}$, $i = 1, \ldots, n_x$, are optimal *unknown* (constant) weights that minimize the approximation error over $\mathcal{D}_{\mathrm{c}_x} \times \mathcal{D}_{\mathrm{c}_z}$, $\varepsilon_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^m$, $i = 1, \ldots, n_x$, are modeling errors such that $\sigma_{\max}(\Upsilon(x, z)) \leq \gamma^{-1}$, $(x, z) \in \mathcal{D}_{\mathrm{c}_x} \times \mathcal{D}_{\mathrm{c}_z}$, where $\Upsilon(x, z) \triangleq [\varepsilon_1(x, z), \ldots, \varepsilon_{n_x}(x, z)]$ and $\gamma > 0$, and $\sigma : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^s$ is a given basis function such that each component of $\sigma(\cdot, \cdot)$ takes values between 0 and 1.

Next, defining

$$\varphi(x, z) \triangleq \delta(x, z) - W^{\mathrm{T}}[x \otimes \sigma(x, z)] \quad (7)$$

where $W^{\mathrm{T}} \triangleq [W_1^{\mathrm{T}}, \ldots, W_{n_x}^{\mathrm{T}}] \in \mathbb{R}^{m \times n_x s}$ and $\otimes$ denotes the Kronecker product, it follows from (6) and Cauchy–Schwarz inequality that

$$\begin{aligned}\varphi^{\mathrm{T}}(x, z)\varphi(x, z) &= \left\|\Delta(x, z)x - W^{\mathrm{T}}(x \otimes \sigma(x, z))\right\|^2 \\ &= \|\Delta(x, z)x - \Sigma(x, z)x\|^2 \\ &= \|\Upsilon(x, z)x\|^2 \\ &\leq \gamma^{-2}x^{\mathrm{T}}x, \quad (x, z) \in \mathcal{D}_{\mathrm{c}x} \times \mathcal{D}_{\mathrm{c}z} \quad (8)\end{aligned}$$

where $\Sigma(x, z) \triangleq [W_1^{\mathrm{T}}\sigma(x, z), \ldots, W_{n_x}^{\mathrm{T}}\sigma(x, z)]$. In the case where $\mathcal{G}$ does not possess internal dynamics (i.e., $n_z = 0$), (6) and (8) specialize to

$$\mathrm{col}_i(\Delta(x)) = W_i^{\mathrm{T}}\sigma(x) + \varepsilon_i(x), \quad x \in \mathcal{D}_{\mathrm{c}x}, \quad i = 1, \ldots, n_x \quad (9)$$

and

$$\varphi^{\mathrm{T}}(x)\varphi(x) \leq \gamma^{-2}x^{\mathrm{T}}x, \quad x \in \mathcal{D}_{\mathrm{c}x} \quad (10)$$

respectively. This corresponds to a nonlinear small gain-type norm bounded uncertainty characterization for $\varphi(\cdot)$ (see Fig. 1).

Theorem 2.1: Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (1) and (2) where $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by (3) and (4), respectively, and $\Delta f(\cdot, \cdot)$ belongs to $\mathcal{F}$. Assume there exists a matrix $K \in \mathbb{R}^{m \times n_x}$ such that $A_{\mathrm{s}} \triangleq A + BK$

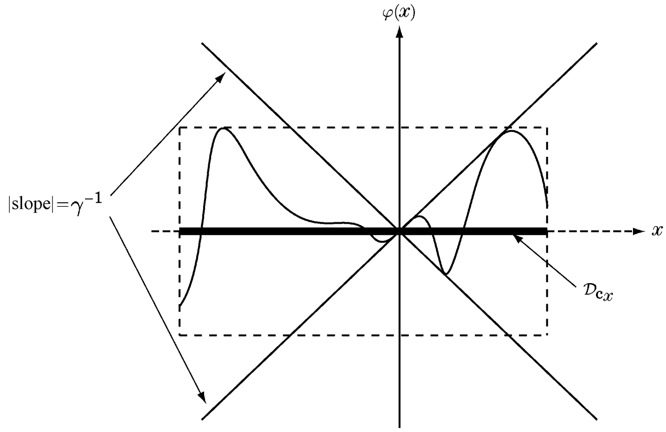

Fig. 1. Visualization of function $\varphi(\cdot)$.

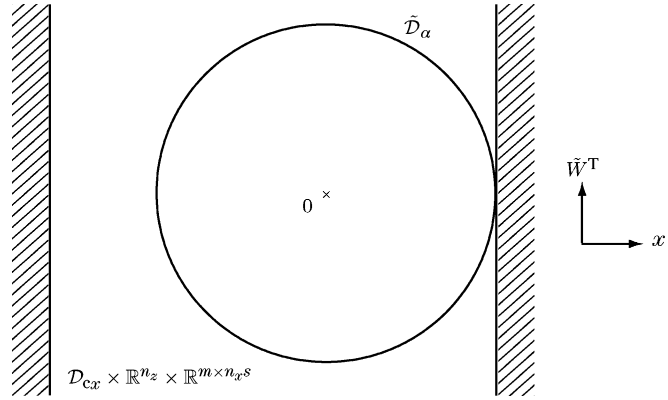

Fig. 2. Visualization of sets used in the proof of Theorem 2.1.

is Hurwitz. Furthermore, for a given $\gamma > 0$, assume there exist positive–definite matrices $P \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_x \times n_x}$ such that

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P + PA_{\mathrm{s}} + \gamma^{-2} PBB^{\mathrm{T}} P + I_{n_x} + R. \tag{11}$$

In addition, assume that (2) is input-to-state stable with $x(t)$ viewed as the input. Finally, let $Q \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{n_x s \times n_x s}$ be positive definite. Then, the neural adaptive feedback control law

$$u(t) = G_{\mathrm{n}}^{-1}(x(t), z(t)) \Big[Kx(t) - \hat{W}^{\mathrm{T}}(t)[x(t) \otimes \sigma(x(t), z(t))] \Big] \tag{12}$$

where $\hat{W}^{\mathrm{T}}(t) \in \mathbb{R}^{m \times n_x s}$, $t \geq 0$, and $\sigma : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^s$, with update law

$$\dot{\hat{W}}^{\mathrm{T}}(t) = QB^{\mathrm{T}} Px(t)[x(t) \otimes \sigma(x(t), z(t))]^{\mathrm{T}} Y, \quad \hat{W}^{\mathrm{T}}(0) = \hat{W}_0^{\mathrm{T}} \tag{13}$$

guarantees that there exists a positively invariant set $\mathcal{D}_\alpha \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{m \times n_x s}$, with $(0, 0, W^{\mathrm{T}}) \in \mathcal{D}_\alpha$, such that the solution $(x(t), z(t), \hat{W}^{\mathrm{T}}(t)) \equiv (0, 0, W^{\mathrm{T}})$ of the closed-loop system given by (1), (2), (12), and (13) is Lyapunov stable and $(x(t), z(t)) \to (0, 0)$ as $t \to \infty$ for all $\Delta f(\cdot, \cdot) \in \mathcal{F}$ and $(x_0, z_0, \hat{W}_0^{\mathrm{T}}) \in \mathcal{D}_\alpha$.

Proof: First, note that with $u(t)$, $t \geq 0$, given by (12), it follows from (1), (3), and (4) that

$$\dot{x}(t) = Ax(t) + \Delta f(x(t), z(t)) + BKx(t) - B\hat{W}^{\mathrm{T}}(t) \times [x(t) \otimes \sigma(x(t), z(t))], \quad x(0) = x_0; \quad t \geq 0 \tag{14}$$

or, equivalently, using (7)

$$\dot{x}(t) = A_{\mathrm{s}} x(t) + B \Big[\varphi(x(t), z(t)) - \tilde{W}^{\mathrm{T}}(t)[x(t) \otimes \sigma(x(t), z(t))] \Big], \quad x(0) = x_0; \quad t \geq 0 \tag{15}$$

where $\tilde{W}^{\mathrm{T}}(t) \triangleq \hat{W}^{\mathrm{T}}(t) - W^{\mathrm{T}}$. To show Lyapunov stability of the closed-loop system (2), (13), and (15), consider the partial Lyapunov function candidate

$$V(x, z, \tilde{W}^{\mathrm{T}}) = x^{\mathrm{T}} Px + \mathrm{tr} Q^{-1} \tilde{W}^{\mathrm{T}} Y^{-1} \tilde{W}. \tag{16}$$

Note that (16) satisfies $\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|)$, $x_1 \in \mathbb{R}^{n_x(1+ms)}$, and $x_2 \in \mathbb{R}^{n_z}$, with $x_1 = [x^{\mathrm{T}}, (\mathrm{vec}\, \tilde{W})^{\mathrm{T}}]^{\mathrm{T}}$, $x_2 = z$, and $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^2$, where $\|x_1\|^2 = x^{\mathrm{T}} Px + \mathrm{tr} Q^{-1} \tilde{W}^{\mathrm{T}} Y^{-1} \tilde{W}$. Now, letting $x(t)$, $t \geq 0$, denote the solution to (15) and using (8), (11), and (13), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}&\left(x(t), z(t), \tilde{W}^{\mathrm{T}}(t)\right) \\
&= 2x^{\mathrm{T}}(t) P \Big[A_{\mathrm{s}} x(t) + B \Big[\varphi(x(t), z(t)) - \tilde{W}^{\mathrm{T}}(t)[x(t) \otimes \sigma(x(t), z(t))] \Big] \Big] \\
&\quad + 2\mathrm{tr} Q^{-1} \tilde{W}^{\mathrm{T}}(t) Y^{-1} \dot{\hat{W}}(t) \\
&= -x^{\mathrm{T}}(t) \left(R + \gamma^{-2} PBB^{\mathrm{T}} P + I_{n_x} \right) x(t) + 2x^{\mathrm{T}}(t) PB \\
&\quad \times \Big[\varphi(x(t), z(t)) - \tilde{W}^{\mathrm{T}}(t)[x(t) \otimes \sigma(x(t), z(t))] \Big] \\
&\quad + 2\mathrm{tr} \tilde{W}^{\mathrm{T}}(t) \left(B^{\mathrm{T}} Px(t)[x(t) \otimes \sigma(x(t), z(t))]^{\mathrm{T}} \right)^{\mathrm{T}} \\
&= -x^{\mathrm{T}}(t) Rx(t) - x^{\mathrm{T}}(t) \left(\gamma^{-2} PBB^{\mathrm{T}} P + I_{n_x} \right) x(t) \\
&\quad + 2x^{\mathrm{T}}(t) PB\varphi(x(t), z(t)) \\
&\leq -x^{\mathrm{T}}(t) Rx(t) - \left[\gamma^{-1} B^{\mathrm{T}} Px(t) + \gamma \varphi(x(t), z(t)) \right]^{\mathrm{T}} \\
&\quad \times \left[\gamma^{-1} B^{\mathrm{T}} Px(t) + \gamma \varphi(x(t), z(t)) \right] \\
&\leq -x^{\mathrm{T}}(t) Rx(t) \\
&\leq 0, \qquad t \geq 0.
\end{aligned} \tag{17}$$

Next, let

$$\tilde{\mathcal{D}}_\alpha \triangleq \left\{ (x, z, \tilde{W}^{\mathrm{T}}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{m \times n_x s} : V(x, z, \tilde{W}^{\mathrm{T}}) \leq \alpha \right\} \tag{18}$$

where α is the maximum value such that $\tilde{\mathcal{D}}_\alpha \subseteq \mathcal{D}_{\mathrm{c}_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{m \times n_x s}$ (see Fig. 2). Now, since $\dot{V}(x(t), z(t), \tilde{W}^{\mathrm{T}}(t)) \leq 0$ for all $(x(t), z(t), \tilde{W}^{\mathrm{T}}(t)) \in \tilde{\mathcal{D}}_\alpha$ and $t \geq 0$, it follows that $\tilde{\mathcal{D}}_\alpha$ is positively invariant. Furthermore, it follows from [15, Th. 1] that the solution $(x(t), z(t), \hat{W}^{\mathrm{T}}(t)) \equiv (0, 0, W^{\mathrm{T}})$ to (2), (13), and (15) is Lyapunov stable with respect to x and $\hat{W}^{\mathrm{T}}$ (uniformly in z_0) for all $\Delta f(\cdot,\cdot) \in \mathcal{F}$ and $(x_0, z_0, \hat{W}_0) \in \tilde{\mathcal{D}}_\alpha$. In addition, since $R > 0$, it follows from [15, Th. 2] that $x(t) \to 0$ as $t \to \infty$.

Next, since (2) is input-to-state stable with $x(t)$ viewed as the input, it follows from [18, Th. 1] that there exists a continuously differentiable, radially unbounded positive–definite function $V_z : \mathbb{R}^{n_z} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$V_z'(z) f_z(x, z) \le -\gamma_1(\|z\|), \qquad \|z\| \ge \gamma_2(\|x\|). \tag{19}$$

Since $\|x(t)\|$ is bounded for all $t \ge 0$, it follows that the set given by

$$\mathcal{D}_z \triangleq \left\{ z \in \mathbb{R}^{n_z} : V_z(z) \le \max_{\|z\| = \gamma_2\left(\sup_{t \ge 0} \|x(t)\|\right)} V_z(z) \right\} \tag{20}$$

is also positively invariant as long as[1] $\mathcal{D}_z \subset \mathcal{D}_{c_z}$. Now, since $\tilde{\mathcal{D}}_\alpha$ and $\mathcal{D}_z$ are positively invariant, it follows that

$$\mathcal{D}_\alpha \triangleq \Big\{ (x, z, \hat{W}^{\mathrm{T}}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{m \times n_x s} : (x, z, \hat{W}^{\mathrm{T}} - W^{\mathrm{T}}) \in \tilde{\mathcal{D}}_\alpha, z \in \mathcal{D}_z \Big\} \tag{21}$$

is also positively invariant. Furthermore, since (2) is input-to-state stable with $x(t)$ viewed as the input, it follows from Proposition 6.1 (see the Appendix) that the solution $(x(t), z(t), \hat{W}^{\mathrm{T}}(t)) \equiv (0, 0, W^{\mathrm{T}})$ to (2), (13), and (15) is Lyapunov stable and $x(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$ for all $\Delta f(\cdot,\cdot) \in \mathcal{F}$ and $(x_0, z_0, \hat{W}_0) \in \mathcal{D}_\alpha$. □

Remark 2.1: Note that the conditions in Theorem 2.1 imply partial asymptotic stability, that is, the solution $(x(t), z(t), \hat{W}^{\mathrm{T}}(t)) \equiv (0, 0, W^{\mathrm{T}})$ of the overall closed-loop system is Lyapunov stable and $(x(t), z(t)) \to (0, 0)$ as $t \to \infty$. Hence, it follows from (13) that $\dot{\hat{W}}^{\mathrm{T}}(t) \to 0$ as $t \to \infty$.

Remark 2.2: Since the partial Lyapunov function (16) used in the proof of Theorem 2.1 is a class $\mathcal{K}_\infty$ function with respect to x and $\tilde{W}$, the assumption $\mathcal{D}_z \subset \mathcal{D}_{c_z}$ invoked in the proof of Theorem 2.1 is automatically satisfied in the case where the NN approximation holds in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. Furthermore, in this case, the control law (12) ensures global asymptotic stability with respect to x and z. However, the existence of a global NN approximator for an uncertain nonlinear map cannot, in general, be established. Hence, as is common in the NN literature, for a given arbitrarily large compact set $\mathcal{D}_{c_x} \times \mathcal{D}_{c_z} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, we assume that there exists an approximator for the unknown nonlinear map up to a desired accuracy in the sense of (8). This assumption ensures that there exists a nontrivial Lyapunov level set such that $\mathcal{D}_z \subset \mathcal{D}_{c_z}$. In the case where $\Delta(\cdot,\cdot)$ is continuous on $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, it follows from the Stone–Weierstrass theorem that $\Delta(\cdot,\cdot)$ can be approximated over an arbitrarily large compact set $\mathcal{D}_{c_x} \times \mathcal{D}_{c_z}$. In this case, our neuroadaptive controller guarantees semiglobal partial asymptotic stability.

[1]See Remark 2.2.

Remark 2.3: Note that the neuroadaptive controller (12) and (13) can be constructed to guarantee partial asymptotic stability using standard *linear* H_∞ theory. Specifically, it follows from standard H_∞ theory [19] that $\|G(s)\|_\infty < \gamma$, where $G(s) = E(sI_n - A_{\mathrm{s}})^{-1}B$ and E is such that $E^{\mathrm{T}}E = I_{n_x} + R$, if and only if there exists a positive–definite matrix P satisfying the bounded real Riccati equation (11). It is well known that (11) has a positive–definite solution if and only if the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A_{\mathrm{s}} & \gamma^{-2}BB^{\mathrm{T}} \\ -E^{\mathrm{T}}E & -A_{\mathrm{s}}^{\mathrm{T}} \end{bmatrix} \tag{22}$$

has no purely imaginary eigenvalues. It is important to note that $\gamma > 0$, which characterizes the approximation error over $\mathcal{D}_{cx} \times \mathcal{D}_{cz}$, can be made arbitrarily large by taking a large number of basis functions in the parameterization of the uncertainty $\Delta(\cdot,\cdot)$. In this case, we can construct the weight update law (13) by solving the linear matrix inequality

$$\begin{bmatrix} A_{\mathrm{s}}^{\mathrm{T}}P + PA_{\mathrm{s}} + I_{n_x} & PB \\ B^{\mathrm{T}}P & -\gamma^2 I_m \end{bmatrix} < 0 \tag{23}$$

for P, with $A_{\mathrm{s}} = A + BK$ and the largest attainable value for γ.

It is important to note that the adaptive control law (12) and (13) does not require the explicit knowledge of the optimal weighting matrix W nor specific structure on the nonlinear dynamics $f_x(x, z)$ and $f_z(x, z)$ to apply Theorem 2.1. However, if (1) is in normal form [20] with (2) being input-to-state stable with x viewed as the input, then we can always construct a neuroadaptive control law *without* requiring knowledge of the system dynamics $f_x(x, z)$ and $f_z(x, z)$ while the assumption (3) is guaranteed to be satisfied. To see this, assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{\mathrm{u}i}(q(t), z(t)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(q(t), z(t))\, u_j(t), \quad t \ge 0, \quad i = 1, \dots, m \tag{24}$$

$$\dot{z}(t) = f_z(q(t), z(t)), \qquad z(0) = z_0 \tag{25}$$

where $q = [q_1, \dots, q_1^{(r_1-1)}, \dots, q_m, \dots, q_m^{(r_m-1)}]^{\mathrm{T}}$, $q(0) = q_0$, $q_i^{(r_i)}$ denotes the r_ith derivative of q_i, and r_i denotes the relative degree with respect to the output q_i. Here, we assume that the square matrix function $G_{\mathrm{s}}(q, z)$ composed of the entries $G_{\mathrm{s}(i,j)}(q, z)$, $i, j = 1, \dots, m$, is such that $\det G_{\mathrm{s}}(q, z) \ne 0$, $(q, z) \in \mathbb{R}^{\hat{r}} \times \mathbb{R}^{n_z}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (24) and $\hat{r} = n_x$. Furthermore, we assume that $f_{\mathrm{u}_i}(\cdot, z)$ is continuously differentiable on $\mathbb{R}^{n_x}$ for each $z \in \mathbb{R}^{n_z}$, $f_{\mathrm{u}_i}(x, \cdot)$ is continuous on $\mathbb{R}^{n_z}$ for each $x \in \mathbb{R}^{n_x}$, and $f_{\mathrm{u}_i}(0, \cdot) = 0$. In addition, we assume that the dynamics given by (25) is input-to-stable with q viewed as the input.

Next, define $x_i \triangleq [q_i, \dots, q_i^{(r_i-2)}]^{\mathrm{T}}$, $i = 1, \dots, m$, $x_{m+1} \triangleq [q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)}]^{\mathrm{T}}$, and $x \triangleq [x_1^{\mathrm{T}}, \dots, x_{m+1}^{\mathrm{T}}]^{\mathrm{T}}$, so that (24) can be described by (1) with

$$A = \begin{bmatrix} A_0 \\ 0_{m \times n_x} \end{bmatrix} \quad \Delta f(x, z) = \begin{bmatrix} 0_{(n_x - m) \times 1} \\ f_{\mathrm{u}}(x, z) \end{bmatrix}$$

$$G(x, z) = \begin{bmatrix} 0_{(n_x - m) \times m} \\ G_{\mathrm{s}}(x, z) \end{bmatrix}$$

where $A_0 \in \mathbb{R}^{(n_x-m)\times n_x}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [21], $f_\mathrm{u} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is an unknown function and satisfies $f_\mathrm{u}(0,\cdot) = 0$, and $G_\mathrm{s} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m\times m}$. Note that $\Delta f(\cdot,\cdot) \in \mathcal{F}$ with $B = [0_{m\times(n_x-m)}, I_m]^\mathrm{T}$ and $\delta(x,z) = f_\mathrm{u}(x,z)$. In this case, $G_\mathrm{n}(x,z) \equiv G_\mathrm{s}(x,z)$. Furthermore, since A is in multivariable controllable canonical form, we can always construct K such that $A+BK$ is Hurwitz.

III. Stable Neuroadaptive Control for Discrete-Time Nonlinear Uncertain Systems

In this section, we develop a similar framework to the framework presented in Section II for *discrete-time* nonlinear uncertain systems. Specifically, consider the controlled nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$x(k+1) = f(x(k)) + G(x(k))\,u(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \tag{26}$$

where $x(k) \in \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \overline{\mathbb{Z}}_+$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and satisfies $f(0) = 0$, and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is continuous. For simplicity of exposition, here we assume that $\mathcal{G}$ does not possess internal dynamics, and hence, a discrete-time subsystem analogous to (2) is not considered. The case where $\mathcal{G}$ possesses input-to-state stable internal dynamics can be handled as in the continuous-time case discussed in Section II using partial stability theory for discrete-time systems.

In this section, we assume that $f(\cdot)$ is an unknown function and $f(\cdot)$ and $G(\cdot)$ are given by

$$f(x) = Ax + \Delta f(x) \tag{27}$$
$$G(x) = BG_\mathrm{n}(x) \tag{28}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known matrices, $G_\mathrm{n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a known matrix function such that $\det G_\mathrm{n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $\Delta f : \mathbb{R}^n \to \mathbb{R}^n$ is an uncertain function belonging to the uncertainty set $\mathcal{F}$ given by

$$\mathcal{F} = \{\Delta f : \mathbb{R}^n \to \mathbb{R}^n : \Delta f(0) = 0, \Delta f(x) = B\delta(x), x \in \mathbb{R}^n\} \tag{29}$$

where $\delta : \mathbb{R}^n \to \mathbb{R}^m$ is an uncertain continuously differentiable function such that $\delta(0) = 0$. As discussed in Section II, since $\delta(x)$ is continuously differentiable and $\delta(0) = 0$, it follows that there exists a continuous matrix function $\Delta : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ such that $\delta(x) = \Delta(x)x$, $x \in \mathbb{R}^n$. Furthermore, we assume that the continuous matrix function $\Delta(\cdot)$ can be approximated over a compact set $\mathcal{D}_\mathrm{c} \subset \mathbb{R}^n$ by a linear in the parameters NN up to a desired accuracy so that (9) and (10) hold for a given $\gamma > 0$ with $\mathcal{D}_{\mathrm{c}_x}$ and n_x replaced by $\mathcal{D}_\mathrm{c}$ and n, respectively, and where $\varphi(x)$ is given by

$$\varphi(x) \triangleq \delta(x) - W^\mathrm{T}[x \otimes \sigma(x)]. \tag{30}$$

Theorem 3.1: Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (26) where $f(\cdot)$ and $G(\cdot)$ are given by (27) and (28), respectively, and $\Delta f(\cdot)$ belongs to $\mathcal{F}$. Assume there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $A_\mathrm{s} \triangleq A + BK$ is Schur. Furthermore, for a given $\gamma > 0$, assume there exist positive–definite matrices $P \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} P = {} & A_\mathrm{s}^\mathrm{T}PA_\mathrm{s} + A_\mathrm{s}^\mathrm{T}PB[\gamma^2 I_m - B^\mathrm{T}PB]^{-1} \\ & \times B^\mathrm{T}PA_\mathrm{s} + (1+\alpha+\beta)I_n + R \end{aligned} \tag{31}$$

$$\gamma^2 I_m - B^\mathrm{T}PB > 0 \tag{32}$$

where $\alpha > 0$ and, for all $x \in \mathbb{R}^n$, β satisfies

$$\begin{aligned} \beta \geq {} & \max\{c, n/\lambda_{\min}(P)\}\,\gamma^{-2} \\ & \times \lambda_{\max}\left(\frac{2}{\alpha}B^\mathrm{T}PA_\mathrm{s}A_\mathrm{s}^\mathrm{T}PB + \frac{2}{\alpha\gamma^2}(B^\mathrm{T}PB)^2 + B^\mathrm{T}PB\right) \\ & \times \frac{1 + x^\mathrm{T}Px}{c + \tilde{\sigma}^\mathrm{T}(x)\tilde{\sigma}(x)} \end{aligned} \tag{33}$$

where $\tilde{\sigma}(x) \triangleq x \otimes \sigma(x)$ and $c > 0$. Then, the neural adaptive feedback control law

$$u(k) = G_\mathrm{n}^{-1}(x(k))\left[Kx(k) - \hat{W}^\mathrm{T}(k)[x(k) \otimes \sigma(x(k))]\right] \tag{34}$$

where $\hat{W}^\mathrm{T}(k) \in \mathbb{R}^{m\times ns}$, $k \in \overline{\mathbb{Z}}_+$, and $\sigma : \mathbb{R}^n \to \mathbb{R}^s$, with update law

$$\begin{aligned} \hat{W}^\mathrm{T}(k+1) = {} & \hat{W}^\mathrm{T}(k) + \frac{1}{c + \sigma^\mathrm{T}(x(k))\sigma(x(k))\|x(k)\|^2}B^\dagger \\ & \times [x(k+1) - A_\mathrm{s}x(k)][x(k) \otimes \sigma(x(k))]^\mathrm{T}, \\ \hat{W}^\mathrm{T}(0) = {} & \hat{W}_0^\mathrm{T} \end{aligned} \tag{35}$$

guarantees that there exists a positively invariant set $\mathcal{D}_\alpha \subset \mathbb{R}^n \times \mathbb{R}^{m\times ns}$, with $(0, W^\mathrm{T}) \in \mathcal{D}_\alpha$, such that the solution $(x(k), \hat{W}^\mathrm{T}(k)) \equiv (0, W^\mathrm{T})$ of the closed-loop system given by (26), (34), and (35) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $\Delta f(\cdot) \in \mathcal{F}$ and $(x_0, \hat{W}_0^\mathrm{T}) \in \mathcal{D}_\alpha$.

Proof: First, note that with $u(k)$, $k \in \overline{\mathbb{Z}}_+$, given by (34) it follows from (26)–(28) that

$$\begin{aligned} x(k+1) = {} & Ax(k) + \Delta f(x(k)) + BKx(k) - B\hat{W}^\mathrm{T}(k) \\ & \times [x(k) \otimes \sigma(x(k))], \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \end{aligned} \tag{36}$$

or, equivalently, using (30)

$$\begin{aligned} x(k+1) = {} & A_\mathrm{s}x(k) + B\left[\varphi(x(k)) - \tilde{W}^\mathrm{T}(k)[x(k) \otimes \sigma(x(k))]\right], \\ & x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \end{aligned} \tag{37}$$

where $\tilde{W}^\mathrm{T}(k) \triangleq \hat{W}^\mathrm{T}(k) - W^\mathrm{T}$. Now, adding and subtracting W^T to and from (35) and using (37) it follows that

$$\begin{aligned} \tilde{W}^\mathrm{T}(k+1) = {} & \tilde{W}^\mathrm{T}(k) + \frac{1}{c + \sigma^\mathrm{T}(x(k))\sigma(x(k))\|x(k)\|^2}B^\dagger B \\ & \times \left[\varphi(x(k)) - \tilde{W}^\mathrm{T}(k)\tilde{\sigma}(x(k))\right]\tilde{\sigma}^\mathrm{T}(x(k)) \\ = {} & \tilde{W}^\mathrm{T}(k) + \frac{1}{c + \tilde{\sigma}^\mathrm{T}(x(k))\tilde{\sigma}(x(k))} \\ & \times \left[\varphi(x(k)) - \tilde{W}^\mathrm{T}(k)\tilde{\sigma}(x(k))\right]\tilde{\sigma}^\mathrm{T}(x(k)). \end{aligned} \tag{38}$$

To show Lyapunov stability of the closed-loop system (35) and (37) consider the Lyapunov function candidate

$$V(x,\tilde{W}^{\mathrm{T}}) = \ln(1+x^{\mathrm{T}}Px) + a\mathrm{tr}\tilde{W}\tilde{W}^{\mathrm{T}} \tag{39}$$

where

$$a = \max\{c, n/\lambda_{\min}(P)\} \times \lambda_{\max}\left(\frac{2}{\alpha}B^{\mathrm{T}}PA_{\mathrm{s}}A_{\mathrm{s}}^{\mathrm{T}}PB + \frac{2}{\alpha\gamma^2}(B^{\mathrm{T}}PB)^2 + B^{\mathrm{T}}PB\right). \tag{40}$$

Note that $V(0,0)=0$ and, since P is positive definite and $a>0$, $V(x,\tilde{W}^{\mathrm{T}})>0$ for all $(x,\tilde{W}^{\mathrm{T}})\neq(0,0)$. Now, letting $x(k)$, $k\in\overline{\mathbb{Z}}_+$, denote the solution to (37) and using (10), (31), and (35), it follows that the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}
&\Delta V\left(x(k),\tilde{W}^{\mathrm{T}}(k)\right)\\
&\quad\triangleq V\left(x(k+1),\tilde{W}^{\mathrm{T}}(k+1)\right) - V\left(x(k),\tilde{W}^{\mathrm{T}}(k)\right)\\
&\quad= \ln\Bigg(1+\left(A_{\mathrm{s}}x(k)+B\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\right)^{\mathrm{T}}\\
&\qquad\times P\left(A_{\mathrm{s}}x(k)+B\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\right)\Bigg)\\
&\qquad+a\mathrm{tr}\Bigg(\tilde{W}^{\mathrm{T}}(k)+\frac{1}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\tilde{\sigma}^{\mathrm{T}}(x(k))\Bigg)^{\mathrm{T}}\\
&\qquad\times\Bigg(\tilde{W}^{\mathrm{T}}(k)+\frac{1}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\times\tilde{\sigma}^{\mathrm{T}}(x(k))\Bigg)\\
&\qquad-\ln\left(1+x^{\mathrm{T}}(k)Px(k)\right)-a\mathrm{tr}\tilde{W}(k)\tilde{W}^{\mathrm{T}}(k)\\
&\quad=\ln\Bigg(1+\Big[x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}}x(k)+2x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB\varphi(x(k))\\
&\qquad-2x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\\
&\qquad+\varphi^{\mathrm{T}}(x(k))BPB\varphi(x(k))\\
&\qquad-2\varphi^{\mathrm{T}}(x(k))BPB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\\
&\qquad+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)BPB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\\
&\qquad-x^{\mathrm{T}}(k)Px(k)\Big]\times\left[1+x^{\mathrm{T}}(k)Px(k)\right]^{-1}\Bigg)\\
&\qquad+a\mathrm{tr}\tilde{W}(k)\tilde{W}^{\mathrm{T}}(k)+\frac{2a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\mathrm{tr}W(k)\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\times\tilde{\sigma}^{\mathrm{T}}(x(k))\\
&\qquad+\frac{a}{\left(c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))\right)^2}\mathrm{tr}\tilde{\sigma}(x(k))\\
&\qquad\times\left[\varphi^{\mathrm{T}}(x(k))-\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\right]\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\times\tilde{\sigma}^{\mathrm{T}}(x(k))\\
&\qquad-a\mathrm{tr}\tilde{W}(k)\tilde{W}^{\mathrm{T}}(k)\\
&\quad\le\Bigg[-x^{\mathrm{T}}(k)\Big((1+\alpha+\beta)I_n+R+A_{\mathrm{s}}^{\mathrm{T}}PB\\
&\qquad\times[\gamma^2I_m-B^{\mathrm{T}}PB]^{-1}B^{\mathrm{T}}PA_{\mathrm{s}}\Big)x(k)\\
&\qquad+2x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB\varphi(x(k))-2x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\\
&\qquad\times\tilde{\sigma}(x(k))+\varphi^{\mathrm{T}}(x(k))B^{\mathrm{T}}PB\varphi(x(k))\\
&\qquad-2\varphi^{\mathrm{T}}(x(k))B^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\\
&\qquad+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)B^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\Bigg]\\
&\qquad\times\left[1+x^{\mathrm{T}}(k)Px(k)\right]^{-1}+\frac{2a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\mathrm{tr}\tilde{W}(k)\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\tilde{\sigma}^{\mathrm{T}}(x(k))\\
&\qquad+\frac{a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\mathrm{tr}\\
&\qquad\times\left[\varphi^{\mathrm{T}}(x(k))-\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\right]\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\\
&\quad\le\Big[-x^{\mathrm{T}}(k)\left((1+\alpha+\beta)I_n+R\right)x(k)-x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB\\
&\qquad\times[\gamma^2I_m-B^{\mathrm{T}}PB]^{-1}B^{\mathrm{T}}PA_{\mathrm{s}}x(k)\\
&\qquad+x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB(\gamma^2I_m-B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA_{\mathrm{s}}x(k)\\
&\qquad+\varphi^{\mathrm{T}}(x(k))(\gamma^2I_m-B^{\mathrm{T}}PB)\varphi(x(k))\\
&\qquad-2x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\\
&\qquad+\varphi^{\mathrm{T}}(x(k))B^{\mathrm{T}}PB\varphi(x(k))\\
&\qquad-2\varphi^{\mathrm{T}}(x(k))B^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\\
&\qquad+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)B^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\Big]\\
&\qquad\times\left[1+x^{\mathrm{T}}(k)Px(k)\right]^{-1}\\
&\qquad+\frac{2a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\mathrm{tr}\tilde{W}(k)\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]\tilde{\sigma}^{\mathrm{T}}(x(k))\\
&\qquad+\frac{a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\\
&\qquad\times\mathrm{tr}\left[\varphi^{\mathrm{T}}(x(k))-\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\right]\\
&\qquad\times\left[\varphi(x(k))-\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\right]
\end{aligned}\tag{41}$$

where in (41) we used $\ln a-\ln b=\ln(a/b)$ and $\ln(1+c)\le c$ for $a,b>0$ and $c>-1$, respectively, and $(\tilde{\sigma}^{\mathrm{T}}\tilde{\sigma}/(c+\tilde{\sigma}^{\mathrm{T}}\tilde{\sigma}))<1$. Furthermore, note that $\tilde{\sigma}^{\mathrm{T}}(x)\tilde{\sigma}(x)\le nx^{\mathrm{T}}x$.

Now, defining $Q_1\triangleq(2/\alpha)B^{\mathrm{T}}PA_{\mathrm{s}}A_{\mathrm{s}}^{\mathrm{T}}PB$ and $Q_2\triangleq(2/\alpha\gamma^2)(B^{\mathrm{T}}PB)^2$, it follows from (41) that

$$\begin{aligned}
&\Delta V\left(x(k),\tilde{W}^{\mathrm{T}}(k)\right)\\
&\quad\le\Bigg[-x^{\mathrm{T}}(k)(I_n+R)x(k)+x^{\mathrm{T}}(k)x(k)\\
&\qquad-\left[x^{\mathrm{T}}(k),\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\right]\begin{bmatrix}\frac{1}{2}\alpha I_n & A_{\mathrm{s}}^{\mathrm{T}}PB\\ B^{\mathrm{T}}PA_{\mathrm{s}} & Q_1\end{bmatrix}\\
&\qquad\times\begin{bmatrix}x(k)\\ \tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))\end{bmatrix}-\left[\varphi^{\mathrm{T}}(x(k)),\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\right]
\end{aligned}$$

$$\times \begin{bmatrix} \frac{1}{2}\alpha\gamma^2 I_n & B^{\mathrm{T}}PB \\ B^{\mathrm{T}}PB & Q_2 \end{bmatrix} \begin{bmatrix} \varphi(x(k)) \\ \tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k)) \end{bmatrix}$$
$$+ \tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)(Q_1+Q_2)\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))$$
$$\left. + \tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)B^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k)) \right]$$
$$\times \left[1+x^{\mathrm{T}}(k)Px(k)\right]^{-1} - \frac{a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}$$
$$\times \tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\tilde{W}^{\mathrm{T}}(k)\tilde{\sigma}(x(k))$$
$$+ \frac{a}{c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k))}\varphi^{\mathrm{T}}(x(k))\varphi(x(k))$$
$$\le -\frac{x^{\mathrm{T}}(k)Rx(k)}{1+x^{\mathrm{T}}(k)Px(k)}$$
$$- \frac{\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{W}(k)\tilde{R}(x(k))\tilde{W}^{\mathrm{T}}\tilde{\sigma}(x(k))}{(c+\tilde{\sigma}^{\mathrm{T}}(x(k))\tilde{\sigma}(x(k)))(1+x^{\mathrm{T}}(k)Px(k))} \quad (42)$$

where

$$\tilde{R}(x) \triangleq a(1+x^{\mathrm{T}}Px)I_m - (Q_1+Q_2+B^{\mathrm{T}}PB) \times \left(c+\tilde{\sigma}^{\mathrm{T}}(x)\tilde{\sigma}(x)\right)$$
$$\ge a(1+x^{\mathrm{T}}Px)I_m - (Q_1+Q_2+B^{\mathrm{T}}PB)(c+nx^{\mathrm{T}}x)$$
$$\ge 0, \qquad x \in \mathcal{D}_{\mathrm{c}}. \quad (43)$$

Hence, the Lyapunov difference given by (42) yields

$$\Delta V\left(x(k), \tilde{W}^{\mathrm{T}}(k)\right) \le -\frac{x^{\mathrm{T}}(k)Rx(k)}{1+x^{\mathrm{T}}(k)Px(k)} \le 0, \qquad k \in \overline{\mathbb{Z}}_+. \quad (44)$$

Next, let

$$\tilde{\mathcal{D}}_\alpha \triangleq \left\{(x, \tilde{W}^{\mathrm{T}}) \in \mathbb{R}^n \times \mathbb{R}^{m\times ns} : V(x, \tilde{W}^{\mathrm{T}}) \le \alpha\right\} \quad (45)$$

where α is the maximum value such that $\tilde{\mathcal{D}}_\alpha \subseteq \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^{m\times ns}$. Since $\Delta V(x(k), W^{\mathrm{T}}(k)) \le 0$ for all $(x(k), \tilde{W}^{\mathrm{T}}(k)) \in \tilde{\mathcal{D}}_\alpha$ and $k \in \overline{\mathbb{Z}}_+$, it follows that $\tilde{\mathcal{D}}_\alpha$ is positively invariant. Next, since $\tilde{\mathcal{D}}_\alpha$ is positively invariant, it follows that

$$\mathcal{D}_\alpha \triangleq \left\{(x, \hat{W}^{\mathrm{T}}) \in \mathbb{R}^n \times \mathbb{R}^{m\times ns} : (x, \hat{W}^{\mathrm{T}} - W^{\mathrm{T}}) \in \tilde{\mathcal{D}}_\alpha\right\} \quad (46)$$

is also positively invariant. Furthermore, it follows from (44) and (the discrete version of) [15, Th. 2] that the solution $(x(k), \hat{W}^{\mathrm{T}}(k)) \equiv (0, W^{\mathrm{T}})$ to (35) and (37) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $\Delta f(\cdot) \in \mathcal{F}$ and $(x_0, \hat{W}_0) \in \mathcal{D}_\alpha$. □

Remark 3.1: The conditions in Theorem 3.1 imply partial asymptotic stability, that is, the solution $(x(k), \hat{W}^{\mathrm{T}}(k)) \equiv (0, W^{\mathrm{T}})$ of the overall closed-loop system is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$. Hence, it follows from (35) that $\hat{W}^{\mathrm{T}}(k+1) - \hat{W}^{\mathrm{T}}(k) \to 0$ as $k \to \infty$.

Remark 3.2: The neuroadaptive controller (34) and (35) can be constructed to guarantee partial asymptotic stability using standard discrete-time *linear* H_∞ theory. Specifically, it follows from standard discrete-time H_∞ theory [22] that $\|G(z)\|_\infty < \gamma$, where $G(z) \sim \left[\begin{array}{c|c} A_{\mathrm{s}} & B \\ \hline E & 0 \end{array}\right]$ and E is such that $E^{\mathrm{T}}E = (1+\alpha+\beta)I_n + R$, if and only if there exists a positive–definite matrix P satisfying the discrete-time bounded real Riccati equation (31). As in the continuous-time case, $\gamma > 0$ characterizing the approximation error over $\mathcal{D}_{\mathrm{c}}$ can be made arbitrarily large provided that we take a large number of basis functions in the parametrization of the uncertainty $\Delta(\cdot)$. In this case, it can be shown that there always exist α and β such that the conditions (31) and (33) are satisfied. To see this, note that since $\sigma(x)$ is assumed to be a *bounded* basis function of x, $\tilde{\sigma}(x)$ has a linear growth rate, and hence, the function $(1+x^{\mathrm{T}}Px)/(c+\tilde{\sigma}^{\mathrm{T}}(x)\tilde{\sigma}(x))$ in (33) is bounded for all $x \in \mathbb{R}^n$. Now, suppose that for given $\alpha > 0$ and $\beta > 0$, $\gamma = \gamma^*$ satisfies (31). In this case, (31) holds with any $\gamma \ge \gamma^*$ and an appropriate $R > 0$. On the other hand, (33) can be clearly satisfied with sufficiently large γ and arbitrary $x \in \mathbb{R}^n$. Hence, it follows that (31) and (33) are satisfied if the parameter $\gamma > 0$ is assumed to be made arbitrarily large. Finally, note that unlike the continuous-time case, P is not required for constructing the adaptive control law.

IV. Illustrative Numerical Examples

In this section, we present two numerical examples to demonstrate the utility of the proposed neuroadaptive control framework for adaptive stabilization.

Example 4.1: Consider the uncertain controlled Liénard system given by

$$\ddot{q}(t) + c(q(t))\dot{q}(t) + k(q(t)) = bu(t),$$
$$q(0) = q_0, \qquad \dot{q}(0) = \dot{q}_0 \quad (47)$$

where $c : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R} \to \mathbb{R}$ are unknown, continuously differential functions. Note that with $x_1 = q$ and $x_2 = \dot{q}$, (47) can be written in state–space form (1), (2), and (8) with $x = [x_1, x_2]^{\mathrm{T}}$, $z = \varnothing$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\Delta f(x) = [0, -c(x_1)x_2 - k(x_1)]^{\mathrm{T}}$, $B = [0, b]^{\mathrm{T}}$, and $G_{\mathrm{n}}(x) = 1$. Here, we assume that the unknown function $\Delta f(x)$ can be written as $\Delta f(x) = B\delta(x)$, where $\delta(x) = (1/b)[-c(x_1)x_2 - k(x_1)]$ is an unknown, continuously differentiable function. Next, let $K = (1/b)[k_1, k_2]$, where k_1 and k_2 are arbitrary scalars, so that $A_{\mathrm{s}} = A + BK = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$. Now, with the proper choice of k_1 and k_2, it follows from Theorem 2.1 that if there exists $P > 0$ satisfying (11), then the neuroadaptive feedback controller (12) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here, we choose $k_1 = -1$, $k_2 = -1$, $\gamma = 3$, and $R = I_2$, so that P satisfying (11) is given by

$$P = \begin{bmatrix} 3.1586 & 1.0627 \\ 1.0627 & 2.3765 \end{bmatrix}. \quad (48)$$

With $c(x_1) = 2(x_1^4 - 1)$, $k(x_1) = x_1 + \tanh(x_1)$, $b = 3$, $Q = 1$, $Y = 0.1I_{12}$, $\sigma(x) = [1/(1+e^{-a_1x_1}), \ldots, 1/(1+e^{-3a_1x_1}), 1/(1+e^{-a_2x_2}), \ldots, 1/(1+e^{-3a_2x_2})]$, where $a_1 = a_2 = 0.5$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $\hat{W}(0) = 0_{12\times 1}$, Fig. 3 shows the phase portrait of the controlled and uncontrolled system. Note that the neuroadaptive controller is switched on at $t = 10$ s. Fig. 4 shows the state trajectories versus

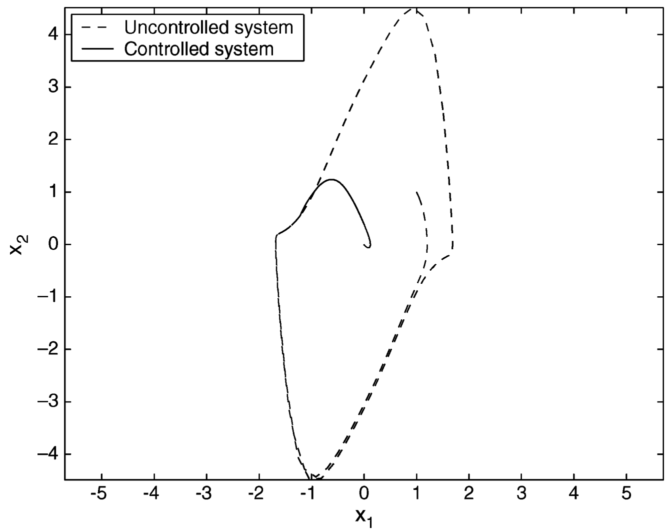

Fig. 3. Phase portrait of controlled and uncontrolled Liénard system.

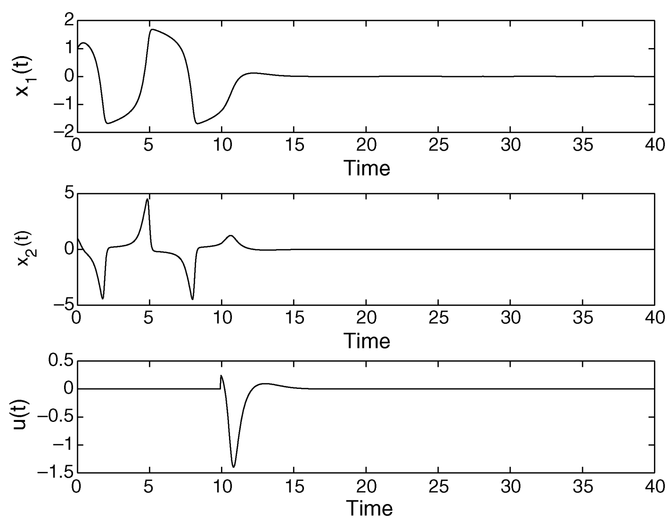

Fig. 4. State trajectories and control signal versus time.

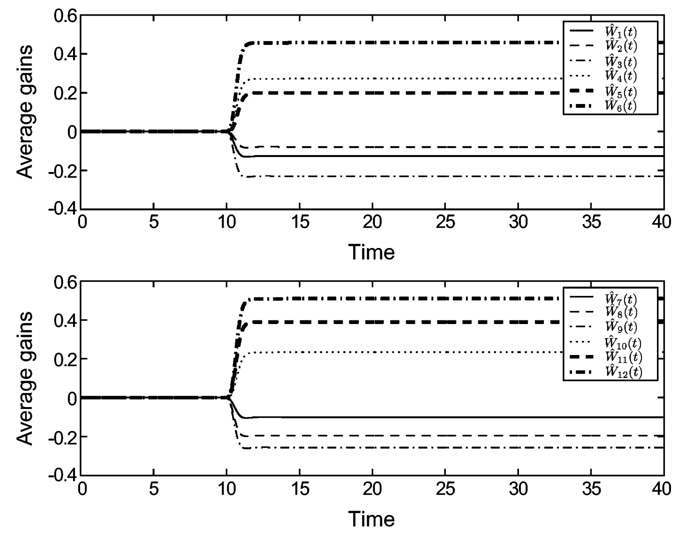

Fig. 5. NN weighting functions versus time.

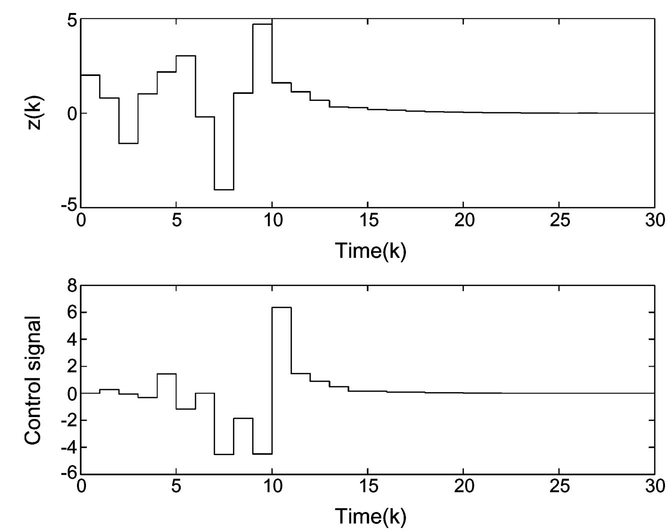

Fig. 6. State trajectory and control signal versus time.

time and the control signal versus time. Finally, Fig. 5 shows the NN weighting functions versus time.

Example 4.2: Consider the nonlinear uncertain discrete-time system given by

$$z(k+2) + f_{\mathrm{u}}(z(k), z(k+1)) = bu(k), \quad z(0) = z_0, \qquad z(1) = z_1 \tag{49}$$

where $z(k) \in \mathbb{R}$, $k \in \overline{\mathbb{Z}}_+$, $u(k) \in \mathbb{R}$, $k \in \overline{\mathbb{Z}}_+$, and $f_{\mathrm{u}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (49) can be written in state–space form (26) with $x = [x_1, x_2]^{\mathrm{T}}$, $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$, $\Delta f(x) = [0, -a_1 x_1 - a_2 x_2 - f_{\mathrm{u}}(x_1, x_2)]^{\mathrm{T}}$, and $G(x) = [0, b]^{\mathrm{T}}$, where $a_1, a_2 \in \mathbb{R}$. Here, we assume that $\Delta f(x)$ is unknown and can be written as $\Delta f(x) = B\delta(x)$, where $\delta(x) = -(1/b)[a_1 x_1 + a_2 x_2 + f_{\mathrm{u}}(x_1, x_2)]$ is an unknown, continuously differentiable function. Next, let $K = (1/b)[k_1, k_2]$, where k_1 and k_2 are arbitrary scalars, so that $A_{\mathrm{s}} = A + BK = \begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}$. Now, with the proper choice of k_1 and k_2, it follows from Theorem 3.1 that if there exists $P > 0$ satisfying (31), then the adaptive feedback controller (34) guarantees that $x(k) \rightarrow 0$ as $k \rightarrow \infty$. Specifically, here, we choose $a_1 = 0$, $a_2 = 0$, $k_1 = 0.1$, $k_2 = 0.1$, $b = 1$, $c = 1$, $\gamma = 18$, $\alpha = 1$, $\beta = 2.8001$, $\sigma(x) = [1, \tanh(\lambda_1 x_1), \ldots, \tanh(6\lambda_1 x_1), \tanh(\lambda_2 x_2), \ldots, \tanh(6\lambda_2 x_2)]^{\mathrm{T}}$, where $\lambda_1 = \lambda_2 = 0.1$, and $R = 0.1999 I_2$ so that P satisfying (31) is given by

$$P = \begin{bmatrix} 5.1057 & 0.1179 \\ 0.1179 & 10.2358 \end{bmatrix}.$$

With $f_{\mathrm{u}}(x_1, x_2) = c_1(x_1^3/(1 + x_1^2)) + c_2 \ln(1 + x_2^2) + c_3 x_1^2$, $c_1 = 1.5$, $c_2 = -0.8$, $c_3 = -0.2$, and initial conditions $x(0) = [2, 0.8]^{\mathrm{T}}$ and $\hat{W}(0) = 0_{26\times 1}$, Fig. 6 shows the state trajectory versus time and the control signal versus time. Finally, Fig. 7 shows the adaptive gain history versus time.

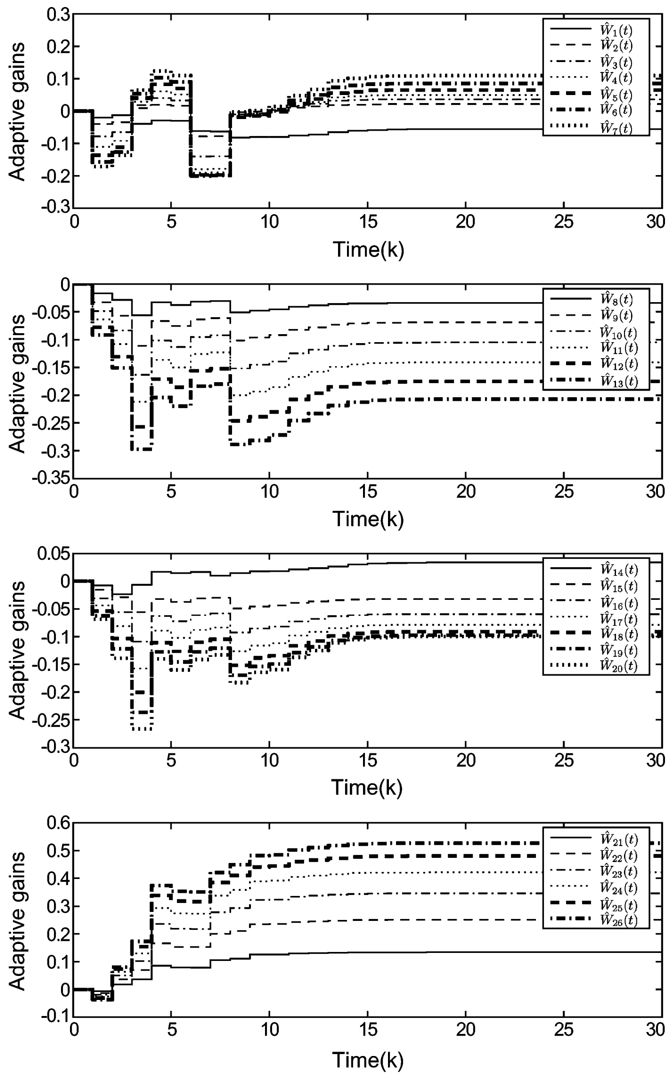

Fig. 7. NN weighting functions versus time.

V. Conclusion

A neuroadaptive control framework for adaptive stabilization of continuous- and discrete-time nonlinear uncertain dynamical systems was developed. In particular, using Lyapunov methods along with robust control techniques and partial stability notions, the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system, that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable internal dynamics of unknown order, the neuroadaptive controllers were constructed without requiring knowledge of the system dynamics other than the fact that the plant dynamics are continuously differentiable. Finally, two illustrative numerical examples were presented to show the utility of the proposed neuroadaptive stabilization scheme.

Appendix

In this appendix, we state and prove a key proposition necessary for proving Theorem 2.1. For this result, recall the definitions of class $\mathcal{K}$ and class $\mathcal{KL}$ functions [10]. Here, we prove the global version of this proposition; the local case is identical except for restricting the domain of analysis.

Proposition 6.1: Consider the nonlinear interconnected dynamical system

$$\dot{x}_1(t) = f_1\left(x_1(t), x_2(t)\right), \qquad x_1(0) = x_{10}, \qquad t \geq 0 \quad (50)$$
$$\dot{x}_2(t) = f_2\left(x_1(t), x_2(t)\right), \qquad x_2(0) = x_{20} \quad (51)$$

where $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(\cdot, x_2)$ is Lipschitz continuous in x_1, and $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that for every $x_1 \in \mathbb{R}^{n_1}$, $f_2(x_1, \cdot)$ is Lipschitz continuous in x_2. If (51) is input-to-state stable with x_1 viewed as the input and (50) and (51) are globally asymptotically stable with respect to x_1 uniformly in x_{20}, then the zero solution $(x_1(t), x_2(t)) \equiv (0, 0)$ of the interconnected dynamical system (50) and (51) is globally asymptotically stable.

Proof: Since (51) is input-to-state stable with x_1 viewed as the input and (50) and (51) are globally asymptotically stable with respect to x_1 uniformly in x_{20}, it follows that there exist class $\mathcal{KL}$ functions $\eta_1(\cdot, \cdot)$ and $\eta_2(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\|x_1(t)\| \leq \eta_1\left(\|x_1(s)\|, t-s\right) \quad (52)$$
$$\|x_2(t)\| \leq \eta_2\left(\|x_2(s)\|, t-s\right) + \gamma\left(\sup_{s \leq \tau \leq t} \|x_1(\tau)\|\right) \quad (53)$$

where $t \geq s \geq 0$. Setting $s = t/2$ in (53) yields

$$\|x_2(t)\| \leq \eta_2\left(\|x_2(t/2)\|, t/2\right) + \gamma\left(\sup_{t/2 \leq \tau \leq t} \|x_1(\tau)\|\right). \quad (54)$$

Next, setting $s = 0$ and replacing t by $t/2$ in (53) yields

$$\|x_2(t/2)\| \leq \eta_2\left(\|x_2(0)\|, t/2\right) + \gamma\left(\sup_{0 \leq \tau \leq t/2} \|x_1(\tau)\|\right). \quad (55)$$

Now, using (52), it follows that

$$\sup_{0 \leq \tau \leq t/2} \|x_1(\tau)\| \leq \eta_1\left(\|x_1(0)\|, 0\right) \quad (56)$$
$$\sup_{t/2 \leq \tau \leq t} \|x_1(\tau)\| \leq \eta_1\left(\|x_1(0)\|, t/2\right). \quad (57)$$

Next, substituting (55)–(57) into (54) and using the fact that $\|x_1(0)\| \leq \|x(0)\|$, $\|x_2(0)\| \leq \|x(0)\|$, and $\|x(t)\| \leq \|x_1(t)\| + \|x_2(t)\|$, $t \geq 0$, where $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$, it follows that

$$\|x(t)\| \leq \eta\left(\|x(0)\|, t\right) \quad (58)$$

where

$$\eta(r, s) = \eta_1(r, s) + \eta_2\left(\eta_2(r, s/2) + \gamma\left(\eta_1(r, 0)\right), s/2\right) + \gamma\left(\eta_1(r, s/2)\right). \quad (59)$$

Finally, since $\eta(\cdot, t)$ is a strictly increasing function with $\eta(0, t) = 0$ and $\eta(r, \cdot)$ is a decreasing function of time such that $\lim_{t \to \infty} \eta(r, t) = 0$, $r > 0$, it follows that the zero solution $x(t) \equiv 0$ of the interconnected dynamical system (50) and (51) is globally asymptotically stable. □

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