Direct Adaptive Control for Discrete-Time Nonlinear Uncertain Dynamical Systems

Wassim M. Haddad¹, Tomohisa Hayakawa¹, and Alexander Leonessa¹

¹School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150
²Department of Ocean Engineering, Florida Atlantic University - Sea Tech, Dania Beach, FL 33004-3023

Abstract

A direct adaptive nonlinear control framework for discrete-time multivariable nonlinear uncertain systems with exogenous bounded disturbances is developed. The adaptive nonlinear controller addresses adaptive stabilization, disturbance rejection, and adaptive tracking. The proposed framework is based on Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to art states associated with the plant. Finally, two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

1. Introduction

In this section we consider the problem of characterizing adaptive feedback control laws for nonlinear uncertain discrete-time systems with exogenous disturbances. Specifically, consider the following controlled nonlinear uncertain discrete-time system \( G \) given by

\[
x(k + 1) = f(x(k)) + G(x(k))u(k) + J(x(k))w(k),
\]

\[x(0) = x_0, \quad k \in \mathcal{N}, \]

where \( x(k) \in \mathbb{R}^n \), \( k \in \mathcal{N}, \) is the state vector, \( u(k) \in \mathbb{R}^m \), \( k \in \mathcal{N}, \) is the control input, \( w(k) \in \mathbb{R}^d \), \( k \in \mathcal{N}, \) is a known bounded disturbance vector such that \(|u(k)|_2 \leq \delta, k \in \mathcal{N}, f : \mathbb{R}^n \to \mathbb{R}^n \) and satisfies \( f(0) = 0, G : \mathbb{R}^n \to \mathbb{R}^{n \times m}, J : \mathbb{R}^n \to \mathbb{R}^{n \times d} \) is a disturbance weighting matrix function with unknown entries. The control input \( u(\cdot) \) in (1) is restricted to the class of admissible controls consisting of measurable functions such that \( u(k) \in \mathbb{R}^m, k \in \mathcal{N}. \)

**Theorem 2.1.** Consider the nonlinear system \( G \) given by (1). Assume there exists a matrix \( K_0 \in \mathbb{R}^{m \times n} \) and functions \( G : \mathbb{R}^n \to \mathbb{R}^{m \times m} \) and \( F : \mathbb{R}^n \to \mathbb{R}^q \) such that \( \det G(x) \neq 0, x \in \mathbb{R}^n \),

\[
F^T(x)F(x) \leq \gamma^2 x^T x, \quad x \in \mathbb{R}^n,
\]

where \( \gamma > 0 \), and the zero solution \( x(k) \equiv 0 \) to

\[
x(k + 1) = f(x(k)) + G(x(k))G^T(x(k))K_0F(x(k))
\]

\[\triangleq f_0(x(k)), \quad x(0) = x_0, \quad k \in \mathcal{N}, \]

is globally asymptotically stable. Furthermore, assume there exists a matrix \( \Psi \in \mathbb{R}^{m \times d} \) such that \( G(x)G(x)\Psi = J(x) \). In addition, assume there exist functions \( V_0 : \mathbb{R}^n \to \mathbb{R}, P_{32} : \mathbb{R}^n \to \mathbb{R}^{1 \times m}, t : \mathbb{R}^n \to \mathbb{R}^t, \) and a nonnegative-definite matrix function \( P_{23} : \mathbb{R}^n \to \mathbb{R}^{m \times m} \) such that

\[
P_{23}(x) \leq \gamma I, \quad x \in \mathbb{R}^n, \quad \gamma > 0, \quad V_0(x) \text{ continuous, positive definite, } V_0(0) = 0, \text{ and, for all } x \in \mathbb{R}^n \text{ and } \bar{u} \in \mathbb{R}^m, \]

\[
V_4(f(x) + G(x)\tilde{G}(x)\bar{u}) = V_4(f(x)) + P_{14}(x)u + \bar{u}^T P_{23}(x)u, \quad 0 \leq V_4(f(x)) - V_4(x) + \epsilon P_{14}(x)P_{14}^T(x), \quad \epsilon \geq 0,
\]

\[
V_4(x) \geq \mu x^T x, \quad (6)
\]

where \( \epsilon \in \mathbb{R} \) and \( \mu \in \mathbb{R} \) are positive constants. Finally, let \( \tilde{x}(\cdot) \triangleq [F^T(x(k)), u^T(k)]^T \) and \( Q \in \mathbb{R}^{m \times m} \) be positive definite such that \( \lambda_{\text{min}}(Q) < 2 \). Then the adaptive feedback control law

\[
u(k) = \tilde{G}(x(k))K(\tilde{x}(k)),
\]

where \( K(\tilde{x}(k)) \in \mathbb{R}^{m \times (s+d)}, k \in \mathcal{N}, \) with update law

\[
K(k + 1) = K(k) - Q\tilde{G}^{-1}(x(k))G^T(x(k))\]

\[
\times [x(k + 1) - f_0(x(k))]^T,
\]

guarantees that the solution \( x(k), K(k) \equiv (0, [K_{23}, -\Psi]) \) of the closed-loop system given by (1), (7), and (8) is Lyapunov stable and \( f(x(k)) \to 0 \) as \( t \to \infty \). If, in addition, \( \tilde{f}(x) \triangleright \tilde{f}(x), x \neq 0, \) then \( x(k) \to 0 \) as \( k \to \infty \) for all \( x_0 \in \mathbb{R}^n. \)
Proof. First, define $\tilde{K}(k) \equiv K(k) - \tilde{K}_g$ and $\tilde{u}(k) \equiv \tilde{K}(k)z(k)$, where $\tilde{K}_g \equiv [K_g, -\Psi]$. Note that with $u(k), k \in \mathcal{N}$, given by (7) it follows from (1) that

$$x(k+1) = f(x(k)) + G(x(k))\tilde{G}(x(k))K(k)\tilde{z}(k) + J(x(k))u(k), \quad x(0) = x_0, \quad k \in \mathcal{N},$$

or, equivalently, using (3) and the fact that $G(x)\tilde{G}(x)\Psi = J(x)$,

$$x(k+1) = f(x(k)) + G(x(k))\tilde{G}(x(k))K(k)\tilde{z}(k) + J(x(k))u(k), \quad x(0) = x_0, \quad k \in \mathcal{N}.$$  \hspace{1cm} (9)

Furthermore, note that by subtracting $\tilde{K}_g$ from both sides of (8) and using (10) it follows that

$$K(k + 1) = K(k) - Q\tilde{G}^{-1}(x(k))G(x(k))\tilde{K}(k)\tilde{z}(k) - [G(x(k))\tilde{G}(x(k))\tilde{K}(k)\tilde{z}(k)]^T(k).$$

To show Lyapunov stability of the closed-loop system (10) and (11), consider the Lyapunov function candidate

$$V(x, K) = \ln(1 + V_0(x)) + a\text{tr}(K - \tilde{K}_g)^TQ(K - \tilde{K}_g),$$

where $a$ is a positive constant. Note that $V(0, \tilde{K}_g) = 0$ and, since $V_0(\cdot)$ and $Q$ are positive definite, $V(x, K) > 0$ for all $(x, K) \neq (0, \tilde{K}_g)$. Furthermore, $V(x, K)$ is radially unbounded. Now, letting $x(k), k \in \mathcal{N}$, denote the solution to (10) and using (4), (5), and (11), it follows that the Lyapunov difference along the closed-loop system trajectories is given by

$$\Delta V(x(k), K(k)) \equiv V(x(k+1), K(k+1)) - V(x(k), K(k)) - a\text{tr}(K(k) - \tilde{K}(k)^TQ^{-1}\tilde{K}(k))$$

with

$$= \ln(1 + V_0(f(x(k)) + G(x(k))\tilde{G}(x(k))\tilde{u}(k)))$$

or

$$= \ln(1 + V_0(f(x(k)) + G(x(k))\tilde{G}(x(k))\tilde{u}(k))) + a\text{tr}(K(k) - \tilde{K}(k)^TQ^{-1}\tilde{K}(k))$$

$$= \ln(1 + V_0(x(k))) + a\text{tr}(K(k) - \tilde{K}(k)^TQ^{-1}\tilde{K}(k))$$

$$\leq \frac{\lambda_{\max}(2I - Q)}{1 + V_0(x(k))} \sum_{i=0}^{k-1} \frac{1}{1 + V_0(x(i))},$$

which proves that the solution $(x(k), K(k)) \equiv (0, \tilde{K}_g)$ to (8) and (10) is Lyapunov stable. Furthermore, it follows from (the discrete-time version of) Theorem 4.4 of [3] that $\ell(x(k)) \to 0$ as $t \to \infty$. Finally, if $\ell(x(k)) > 0$, $x \neq 0$, then $x(k) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

Remark 2.1. Theorem 2.1 is also valid for time-varying uncertain systems $G_t$ of the form

$$x(k+1) = f(k, x(k)) + G(k, x(k))\tilde{G}(x(k))K(k)\tilde{z}(k) + J(k, x(k))u(k), \quad x(0) = x_0, \quad k \in \mathcal{N},$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(k, 0) = 0, k \in \mathcal{N}$, $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and $J : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{m \times d}$. In particular, replacing $F : \mathbb{R}^n \to \mathbb{R}^n$ by $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $F(k, 0) = 0, k \in \mathcal{N}$, and requiring $F^T(k, x(k))F(k, x) \leq \nu I$,
where \( G(k, x) \) is the disturbance matching matrix and \( \Psi \) is the disturbance weight.

Furthermore, since \( G(z)G(z)^T \) is globally asymptotically stable and \( J(z) = D \), the matching condition is satisfied with \( G(z)G(z)^T \) in place of \( G(z) \), and requiring \( G(k, x) \) is globally asymptotically stable and \( J(z) = D \).

To apply Theorem 2.1 to the uncertain system (1) with \( f(z), G(z), \) and \( D \) given by (23) and (24), let \( K_\varepsilon \in \mathbb{R}^{m \times q} \), where \( s = q + r, \) be given by

\[
K_\varepsilon = [\Theta_\varepsilon - \Theta, \Phi_\varepsilon],
\]

where \( \Theta_\varepsilon \in \mathbb{R}^{m \times q} \) and \( \Phi_\varepsilon \in \mathbb{R}^{m \times r} \) are known matrices, and let

\[
F(z) = \begin{bmatrix} f_\varepsilon(z) \\ f_\varepsilon(z) \end{bmatrix},
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a known function and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

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f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
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where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

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f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
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where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

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f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
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where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

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f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
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where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]

where \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and satisfies \( \int f_\varepsilon(z) f_\varepsilon(z)^T \leq \gamma^2 z^T \), \( z \in \mathbb{R}^n, \gamma > 0, \) is an arbitrary function. In this case, it follows that, with \( G(z) = G_\varepsilon^{-1}(z), \)

\[
f_\varepsilon(z) = f(z) + G(z)G(z)^T K_\varepsilon F(z)
\]
can be made globally asymptotically stable. In particular, choosing $\Theta_\nu f(x) + \Phi_\nu f(x) = A\varepsilon$, where $\varepsilon \in \mathbb{R}^{m \times n}$, it follows that (27) has the form $f(x) = A\varepsilon$, where $A = \begin{bmatrix} A_2^T & A_1^T \end{bmatrix}^T$ is in multivariable controllable canonical form. Hence, in the case where $G(x) \equiv B$ is a constant matrix, by choosing $\varepsilon$ such that $A\varepsilon$ is asymptotically stable it follows that for sufficiently small $\varepsilon$ there exists a positive-definite matrix $P$ satisfying the following Riccati-type inequality

$$0 \geq A_2^TPA_2 - P + R + 4eA_1^TPB_1^TPA_1,$$

where $R$ is positive definite. In this case, with Lyapunov function $V_\nu(x) = x^TPx$, (4)-(6) are satisfied with $P_{2\nu}(x) = 2x^TA_2^TPB_1^TPB_2(x) = BT_2PB_1, \mu = \lambda_{\min}(P)$, and hence the adaptive feedback controller (7) with update law (8) guarantees global asymptotic stability of the nonlinear uncertain discrete-time dynamical system (1) where $f(x), G(x)$, and $J(x)$ are given by (23) and (24).

As mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F(x)$ to produce a linear $f(x)$. However, when the system is in normal form, a feedback linearizing function $F(x)$ assures the existence of $V_\nu(x)$ that satisfies the conditions (4) and (5). In particular, choosing $\Theta_\nu = \Phi_\nu = 0$, it follows that

$$G^T(x)f(x) = \begin{bmatrix} 0 & -G_{2\nu}^{-1}(x(x)) \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} x = 0,$$

and hence the update law (8) can be simplified as

$$K(k + 1) = K(k) - QG^{-1}(x(k))G^T(x(k))x(k) + 1 \bar{z}(k).$$

Finally, note that Theorem 2.1 is not restricted to systems with sector-bounded nonlinearities so long as the regressor function $F(x)$ satisfies (2) and we can construct a known function $f(x)$ such that the zero solution $x(k) \equiv 0$ to (3) is globally asymptotically stable.

Next, consider the case where $f(x)$ and $G(x)$ are uncertain. Specifically, we assume that $G_\nu(x)$ is unknown and is parameterized as $G_\nu(x) = B_\nu G(x)$, where $G_\nu : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ is known and satisfies det $G_\nu(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_\nu \in \mathbb{R}^{m \times m}$, with det $B_\nu \neq 0$ and $\sigma_{\min}(B_\nu) < 2$, is an unknown symmetric sign definite matrix but the sign definiteness of $B_\nu$ is known; that is, $B_\nu > 0$ or $B_\nu < 0$. For the statement of the next result define $B_0 \triangleq [0_{m \times (n-m)}, I_m]^T$ for $B_\nu > 0$, and $B_0 \triangleq [0_{m \times (n-m)}, -I_m]^T$ for $B_\nu < 0$.

Corollary 2.1. Consider the nonlinear system $G$ given by (1) with $f(x), G(x)$, and $J(x)$ given by (23) and (24), and $G_\nu(x) = B_\nu G_\nu(x)$, where $B_\nu$ is an unknown symmetric matrix and the sign definiteness of $B_\nu$ is known. Assume there exists a matrix $K_\nu \in \mathbb{R}^{m \times n}$ and a function $F : \mathbb{R}^n \to \mathbb{R}^n$ such that the zero solution $x(k) \equiv 0$ to (3) is globally asymptotically stable and (2) holds. Furthermore, assume there exist functions $V_\nu : \mathbb{R}^n \to \mathbb{R}$, $P_{2\nu} : \mathbb{R}^n \to \mathbb{R}^{m \times n}$, $T : \mathbb{R}^n \to \mathbb{R}^n$, and a nonnegative-definite matrix function $P_{2\nu}(x) \in \mathbb{R}^{m \times m}$, such that $P_{2\nu}(x) \leq \gamma I_m, x \in \mathbb{R}^n, \gamma > 0, V_\nu(x)$ is continuous, positive definite, $V_\nu(0) = 0$, and, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, (4)-(6) hold. Finally, let $\bar{z}(k) \triangleq [F^T(x(k)), w^T(k)]^T$. Then the adaptive feedback control law

$$u(k) = G_{2\nu}^{-1}(x(k))K(k)\bar{z}(k),$$

where $K(k) \in \mathbb{R}^{m \times (q-d)}, k \in \mathbb{N}$, and $\bar{z}(k) \triangleq [F^T(x(k)), w^T(k)]^T$, with update law

$$K(k + 1) = K(k) - B_\nu^T[\dot{x}(x(k)) - f(x(k))]\bar{z}(k),$$

guarantees that the solution $(x(k), K(k))$ is Lyapunov stable and $K(0) = [0, |K_\nu, -\Psi]|$, where $\Psi \in \mathbb{R}^{m \times q}$, of the closed-loop system given by (1), (30), and (31) is Lyapunov stable and $\dot{x}(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. The result is a direct consequence of Theorem 2.1. First, let $G(x) = G_{2\nu}^{-1}(x) \Psi$ and $\Psi = B_\nu^{-1} \Delta$ so that $G(x)\dot{G}(x) = [0_{m \times (n-m)}, B_\nu]D$ and $G(x)\dot{G}(x) \Psi = D$, and let $K_\nu = B_\nu^{-1} \Theta_\nu$. Next, since $Q$ in (8) is an arbitrary positive-definite matrix with $\lambda_{\max}(Q) < 2$, it can be replaced by $[B_\nu] = (B_\nu^2)^{1/2}$, where $(\cdot)^{1/2}$ denotes the (unique) positive-definite square root. Now, since $B_\nu$ is symmetric and sign definite it follows from the Schur decomposition that $B_\nu = UDB_\nu^T$, where $U$ is orthogonal and $D_\nu$ is real diagonal. Hence, $[B_\nu]G_\nu(x)G^T(x) = [0_{m \times (n-m)}, I_m] = B_\nu^2$, where $I_m = I_m$ for $B_\nu > 0$ and $I_m = -I_m$ for $B_\nu < 0$.

3. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed discrete-time adaptive control framework for adaptive stabilization, disturbance rejection, and command following.

Example 3.1. Consider the linear uncertain system given by

$$z(k + 2) + a_1z(k + 1) + a_2z(k) = bu(k) + d \sin 7k,$$

$$z(0) = z_1, z(1) = z_2, k \in \mathbb{N},$$

where $z \in \mathbb{R}, k \in \mathbb{N}, u(k) \in \mathbb{R}, k \in \mathbb{N}$, and $a_0, a_1, a_2, b, d \in \mathbb{R}$ are unknown constants. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k + 1), (32)$ can be written in state space form (1) with $x = [x_1, x_2]^T$, $f(x) = [x_2, -a_2 x_1 - a_1 x_2]^T$, $G(x) = [0, b]^T, J(x) = [0, d]^T$, and $w(k) = \sin 7k$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2]^T$, where $\theta_1$ and $\theta_2$ are unknown constants. Furthermore, we assume that sign $b$ is known and $|b| < 2$. Next, let $G_\nu(x) = 1, F(x) = x$, and $K_\nu = \frac{1}{b} [\theta_1 - \theta_2, \theta_2 - \theta_1]$, where $\theta_1, \theta_2$ are arbitrary scalars, so that

$$f(x) = f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} \theta_1 - \theta_1, \theta_2 - \theta_2 \end{bmatrix} F(x)$$

Now, with the proper choice of $\theta_1$ and $\theta_2$, it follows from Corollary 2.1 that the adaptive feedback controller (30) guarantees that $x(k) \to 0$ as $k \to \infty$. With $\theta_1 = -1, \theta_2 = 0.25, b = 0.4, d = 10, \theta_{01} = -0.02, \theta_{02} = 0.3$, and initial conditions $z(0) = [-1, 3]^T$ and $K(0) = [0, 0, 0]$, Figure 3.1 shows that the phase portrait of the controlled and
uncontrolled system. Note that the adaptive controller is switched on at \( k = 30 \). Figure 3.2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 3.3 shows the adaptive gain history versus time.

**Example 3.2.** Consider the two-degree of freedom uncertain nonlinear system given by

\[
M_s z(k+2) + C_s z(k+1) + K_s z(k) = u(k),
\]

\[
z(0) = z_{00}, \quad z(1) = z_{10}, \quad k \in \mathcal{N}, \quad (33)
\]

where \( z(k) \in \mathbb{R}^2, \ u(k) \in \mathbb{R}^2, \ k \in \mathcal{N}, \ M_s, C_s, K_s \in \mathbb{R}^{2 \times 2} \). Here we assume that \( M_s, C_s, K_s \in \mathbb{R}^{2 \times 2} \).

Let \( r_d(k) \) be a desired command signal and define the error state \( \hat{e}(k) \equiv z(k) - r_d(k) \) so that the error dynamics are given by

\[
M_s \hat{e}(k+2) + C_s \hat{e}(k+1) + K_s \hat{e}(k) = u(k) - M_s r_d(k+2) - C_s r_d(k+1) - K_s r_d(k),
\]

\[
\hat{e}(0) = \hat{e}_0, \quad \hat{e}(1) = \hat{e}_1, \quad k \in \mathcal{N}. \quad (34)
\]

Note that with \( e_1(k) = \hat{e}(k) \) and \( e_2(k) = \hat{e}(k+1) \), (34) can be written in state space form (21) with \( e = [e_1^T, e_2^T]^T, \quad K(k,e) = [e_1^T, -(M_s^{-1} K_s + M_s^{-1} C_s e_2)^T]^T, \quad G(k,e) = [0_{2 \times 2}, M_s^{-1}]^T, \quad J(k,e) = [0_{2 \times 2}, D_e^T]^T \), where \( D_e = [-I_2, -M_s^{-1} C_s, -M_s^{-1} K_s] \) and \( w_k(k) = [r_d^T(k+1), r_d^T(k)]^T \). Note that \( M_s^{-1} \) is symmetric and positive definite but unknown. Next, let \( K_o = M_o \Theta_n + M_s^{-1} K_s, \Theta_n + M_s^{-1} C_s \), where \( \Theta_n, \Theta_o \in \mathbb{R}^{2 \times 2} \), are arbitrary matrices, so that

\[
f_e(e) = \begin{bmatrix} 0_2 & I_2 \\ \Theta_n & \Theta_o \end{bmatrix} e.
\]

Now, with the proper choice of \( \Theta_n \) and \( \Theta_o \), it follows from Corollary 2.1 and Remark 2.2 that the adaptive feedback controller (30) guarantees that \( e(k) \to 0 \) as \( t \to \infty \).

With

\[
M_o = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad C_o = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad K_o = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},
\]

\[
r_d(k) = [\sin 0.5k, 0.5]^T, \quad \Theta_n = \Theta_o = 0_2, \quad \text{and initial conditions } z(0) = [3, -4, -2, 1]^T \quad \text{and } K(0) = 0_{2 \times 10}. \]

Figure 3.4 shows the actual positions and the reference signals versus time and the control signals versus time. Note that the adaptive controller is switched on at \( k = 40 \).

4. Conclusion

A discrete-time direct adaptive nonlinear control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain systems with exogenous bounded disturbances was developed. Using Lyapunov methods the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable zero dynamics, the nonlinear adaptive controllers were constructed without knowledge of the system dynamics. Finally, two illustrative numerical examples were presented to show the utility of the proposed adaptive stabilization and tracking scheme.
Figure 3.4: Positions and control signals versus time

References


