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Direct Adaptive Control for Discrete-Time Nonlinear Uncertain Dynamical Systems

Wassim M. Haddad†, Tomohisa Hayakawa†, and Alexander Leonessa‡

†School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150
‡Department of Ocean Engineering, Florida Atlantic University - Sea Tech, Dania Beach, FL 33004-3023

Abstract

A direct adaptive nonlinear control framework for discrete-time multivariable nonlinear uncertain systems with exogenous bounded disturbances is developed. The adaptive nonlinear controller addresses adaptive stabilization, disturbance rejection, and adaptive tracking. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Finally, two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

1. Introduction

In a recent series of papers [1,2], a direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable continuous-time nonlinear uncertain systems with exogenous bounded amplitude disturbances was developed. In this paper we develop analogous results for discrete-time nonlinear uncertain systems. Specifically, a Lyapunov-based direct adaptive control framework is developed that guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable zero dynamics, the nonlinear discrete-time adaptive controller is constructed *without* requiring knowledge of the system dynamics or system disturbance.

In the paper we use the following standard notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n\times 1$ real column vectors, let $(\,)^{\mathrm{T}}$ denote transpose, let $(\,)^{\dagger}$ denote the Moore-Penrose generalized inverse, and let $\mathcal{N}$ denote the set of nonnegative integers. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M, $\operatorname{tr}(\cdot)$ for the trace operator, and $\ln(\cdot)$ for the natural log operator.

2. Discrete-Time Adaptive Control for Nonlinear Systems with Exogenous Disturbances

In this section we consider the problem of characterizing adaptive feedback control laws for nonlinear uncertain discrete-time systems with exogenous disturbances. Specifically, consider the following controlled nonlinear uncertain discrete-time system $\mathcal{G}$ given by

$$x(k+1) = f(x(k)) + G(x(k))u(k) + J(x(k))w(k), \quad x(0)=x_0, \quad k\in\mathcal{N}, \tag{1}$$

where $x(k)\in\mathbb{R}^n$, $k\in\mathcal{N}$, is the state vector, $u(k)\in\mathbb{R}^m$, $k\in\mathcal{N}$, is the control input, $w(k)\in\mathbb{R}^d$, $k\in\mathcal{N}$, is a known bounded disturbance vector such that $\|w(k)\|_2\le\delta$, $k\in\mathcal{N}$, $f:\mathbb{R}^n\to\mathbb{R}^n$ and satisfies $f(0)=0$, $G:\mathbb{R}^n\to\mathbb{R}^{n\times m}$, and $J:\mathbb{R}^n\to\mathbb{R}^{n\times d}$ is a disturbance weighting matrix function with *unknown* entries. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(k)\in\mathbb{R}^m$, $k\in\mathcal{N}$.

Theorem 2.1. Consider the nonlinear system $\mathcal{G}$ given by (1). Assume there exists a matrix $K_{\mathrm{g}}\in\mathbb{R}^{m\times s}$ and functions $\hat{G}:\mathbb{R}^n\to\mathbb{R}^{m\times m}$ and $F:\mathbb{R}^n\to\mathbb{R}^s$ such that $\det\hat{G}(x)\neq 0$, $x\in\mathbb{R}^n$,

$$F^{\mathrm{T}}(x)F(x)\le\bar{\gamma}^2x^{\mathrm{T}}x, \quad x\in\mathbb{R}^n, \tag{2}$$

where $\bar{\gamma}>0$, and the zero solution $x(k)\equiv 0$ to

$$\begin{aligned} x(k+1) &= f(x(k)) + G(x(k))\hat{G}(x(k))K_{\mathrm{g}}F(x(k)) \\ &\triangleq f_{\mathrm{c}}(x(k)), \quad x(0)=x_0, \quad k\in\mathcal{N}, \end{aligned} \tag{3}$$

is globally asymptotically stable. Furthermore, assume there exists a matrix $\Psi\in\mathbb{R}^{m\times d}$ such that $G(x)\hat{G}(x)\Psi = J(x)$. In addition, assume there exist functions $V_{\mathrm{s}}:\mathbb{R}^n\to\mathbb{R}$, $P_{1\tilde{u}}:\mathbb{R}^n\to\mathbb{R}^{1\times m}$, $\ell:\mathbb{R}^n\to\mathbb{R}^t$, and a nonnegative-definite matrix function $P_{2\tilde{u}}:\mathbb{R}^n\to\mathbb{R}^{m\times m}$ such that $P_{2\tilde{u}}(x)\le\gamma I_m$, $x\in\mathbb{R}^n$, $\gamma>0$, $V_{\mathrm{s}}(\cdot)$ is continuous, positive definite, $V_{\mathrm{s}}(0)=0$, and, for all $x\in\mathbb{R}^n$ and $\tilde{u}\in\mathbb{R}^m$,

$$\begin{aligned} V_{\mathrm{s}}(f(x)+G(x)\hat{G}(x)\tilde{u}) \\ &= V_{\mathrm{s}}(f(x)) + P_{1\tilde{u}}(x)\tilde{u} + \tilde{u}^{\mathrm{T}}P_{2\tilde{u}}(x)\tilde{u}, \end{aligned} \tag{4}$$

$$0 \ge V_{\mathrm{s}}(f_{\mathrm{c}}(x)) - V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x) + \varepsilon P_{1\tilde{u}}(x)P_{1\tilde{u}}^{\mathrm{T}}(x), \tag{5}$$

$$V_{\mathrm{s}}(x) \ge \mu x^{\mathrm{T}}x, \tag{6}$$

where $\varepsilon\in\mathbb{R}$ and $\mu\in\mathbb{R}$ are positive constants. Finally, let $\tilde{x}(k)\triangleq[F^{\mathrm{T}}(x(k)),\, w^{\mathrm{T}}(k)]^{\mathrm{T}}$ and $Q\in\mathbb{R}^{m\times m}$ be positive definite such that $\lambda_{\max}(Q)<2$. Then the adaptive feedback control law

$$u(k)=\hat{G}(x(k))K(k)\tilde{x}(k), \tag{7}$$

where $K(k)\in\mathbb{R}^{m\times(s+d)}$, $k\in\mathcal{N}$, with update law

$$\begin{aligned} K(k+1) &= K(k) - Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k)) \\ &\quad\cdot[x(k+1)-f_{\mathrm{c}}(x(k))]\tilde{x}^{\dagger}(k), \end{aligned} \tag{8}$$

guarantees that the solution $(x(k),K(k))\equiv(0,[K_{\mathrm{g}},\,-\Psi])$ of the closed-loop system given by (1), (7), and (8) is Lyapunov stable and $\ell(x(k))\to 0$ as $t\to\infty$. If, in addition, $\ell^{\mathrm{T}}(x)\ell(x)>0$, $x\neq 0$, then $x(k)\to 0$ as $k\to\infty$ for all $x_0\in\mathbb{R}^n$.

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Proof. First, define $\tilde{K}(k) \triangleq K(k) - \hat{K}_{\rm g}$ and $\tilde{u}(k) \triangleq \tilde{K}(k)\tilde{x}(k)$, where $\hat{K}_{\rm g} \triangleq [K_{\rm g}, -\Psi]$. Note that with $u(k)$, $k \in \mathcal{N}$, given by (7) it follows from (1) that

$$x(k+1) = f(x(k)) + G(x(k))\hat{G}(x(k))K(k)\tilde{x}(k) + J(x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \tag{9}$$

or, equivalently, using (3) and the fact that $G(x)\hat{G}(x)\Psi = J(x)$,

$$\begin{aligned} x(k+1) &= f_{\rm c}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) \\ &\triangleq f_{\rm c}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k), \\ &\qquad x(0) = x_0, \quad k \in \mathcal{N}. \end{aligned} \tag{10}$$

Furthermore, note that by subtracting $\hat{K}_{\rm g}$ from the both sides of (8) and using (10) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k)) \\ &\quad \cdot [G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k)]\tilde{x}^{\dagger}(k) \\ &= \tilde{K}(k) - Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\dagger}(k). \end{aligned} \tag{11}$$

To show Lyapunov stability of the closed-loop system (10) and (11), consider the Lyapunov function candidate

$$V(x, K) = \ln(1 + V_{\rm s}(x)) + a\,{\rm tr}\,(K - \hat{K}_{\rm g})^{\rm T}Q(K - \hat{K}_{\rm g}), \tag{12}$$

where a is a positive constant. Note that $V(0, \hat{K}_{\rm g}) = 0$ and, since $V_{\rm s}(\cdot)$ and Q are positive definite, $V(x, K) > 0$ for all $(x, K) \neq (0, \hat{K}_{\rm g})$. Furthermore, $V(x, K)$ is radially unbounded. Now, letting $x(k)$, $k \in \mathcal{N}$, denote the solution to (10) and using (4), (5), and (11), it follows that the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned} &\Delta V(x(k), K(k)) \\ &\triangleq V(x(k+1), K(k+1)) - V(x(k), K(k)) \\ &= \ln(1 + V_{\rm s}(f_{\rm c}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k))) \\ &\quad + a{\rm tr}\,(\tilde{K}(k) - Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\dagger}(k))^{\rm T}Q^{-1} \\ &\quad \cdot(\tilde{K}(k) - Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\dagger}(k)) \\ &\quad - \ln(1 + V_{\rm s}(x(k))) - a{\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k) \\ &= \ln\Big(1 + \Big[V_{\rm s}(f_{\rm c}(x(k))) + P_{1\tilde{u}}(x(k))\tilde{u}(k) \\ &\quad + \tilde{u}^{\rm T}(k)P_{2\tilde{u}}(x(k))\tilde{u}(k) - V_{\rm s}(x(k))\Big]\,[1 + V_{\rm s}(x(k))]^{-1}\Big) \\ &\quad + a{\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k) - 2a{\rm tr}\,\tilde{K}^{\rm T}(k)\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\dagger}(k) \\ &\quad + a{\rm tr}\,(\tilde{x}(k)\tilde{x}^{\dagger}(k))^{\rm T}\tilde{K}^{\rm T}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\dagger}(k) \\ &\quad - a{\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k) \end{aligned}$$

$\leq [-\ell^{\rm T}(x(k))\ell(x(k)) - \varepsilon P_{1\tilde{u}}(x(k))P^{\rm T}_{1\tilde{u}}(x(k))$

[redacted]

$$\begin{aligned} &\quad + P_{1\tilde{u}}(x(k))\tilde{u}(k) + \gamma\tilde{u}^{\rm T}(k)\tilde{u}(k)]\,[1 + V_{\rm s}(x(k))]^{-1} \\ &\quad - 2a\tilde{x}^{\dagger}(k)\tilde{K}^{\rm T}(k)\tilde{K}(k)\tilde{x}(k) \\ &\quad + a{\rm tr}\,\tilde{x}(k)\tilde{x}^{\dagger}(k)\tilde{K}^{\rm T}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\dagger}(k), \end{aligned} \tag{13}$$

subtracting $\frac{1}{4\varepsilon}\frac{\tilde{u}^{\rm T}(k)\tilde{u}(k)}{1+V_{\rm s}(x(k))}$, $k \in \mathcal{N}$, to and from (13) and collecting terms yields

$$\begin{aligned} &\Delta V(x(k), \tilde{K}(k)) \\ &\leq -\frac{\ell^{\rm T}(x(k))\ell(x(k))}{1 + V_{\rm s}(x(k))} - \frac{1}{1 + V_{\rm s}(x(k))} \\ &\quad \cdot \left[P_{1\tilde{u}}(x(k)),\, \tilde{u}^{\rm T}(k)\right] \begin{bmatrix} \varepsilon I_m & -\frac{1}{2}I_m \\ -\frac{1}{2}I_m & \frac{1}{4\varepsilon}I_m \end{bmatrix} \begin{bmatrix} P^{\rm T}_{1\tilde{u}}(x(k)) \\ \tilde{u}(k) \end{bmatrix} \\ &\quad + \frac{\frac{1}{4\varepsilon}\tilde{u}^{\rm T}(k)\tilde{u}(k) + \gamma\tilde{u}^{\rm T}\tilde{u}}{1 + V_{\rm s}(x(k))} - 2a\tilde{x}^{\dagger}(k)\tilde{K}^{\rm T}(k)\tilde{K}(k)\tilde{x}(k) \\ &\quad + a\tilde{x}^{\dagger}(k)\tilde{K}^{\rm T}(k)Q\tilde{K}(k)\tilde{x}(k)[\tilde{x}^{\dagger}(k)\tilde{x}(k)] \\ &\leq -\frac{\ell^{\rm T}(x(k))\ell(x(k))}{1 + V_{\rm s}(x(k))} \\ &\quad - \frac{\tilde{x}^{\rm T}(k)\tilde{K}^{\rm T}(k)\tilde{R}(x(k), w(k))\tilde{K}(k)\tilde{x}(k)}{\tilde{x}^{\rm T}(k)\tilde{x}(k)(1 + V_{\rm s}(x(k)))}, \end{aligned} \tag{14}$$

where

$$\tilde{R}(x, w) \triangleq a(1 + V_{\rm s}(x))(2I - Q) - \left(\frac{1}{4\varepsilon} + \gamma\right)\tilde{x}^{\rm T}\tilde{x}. \tag{15}$$

Noting that $2I - Q > 0$, since $\lambda_{\max}(Q) < 2$, and taking

$$a \geq \frac{\frac{1}{4\varepsilon} + \gamma}{\lambda_{\min}(2I - Q)} \cdot \max\left\{\delta, \frac{\bar{\gamma}^2}{\mu}\right\},$$

it follows that

$$\begin{aligned} \tilde{R}(x, w) &\geq a(1 + \mu x^{\rm T}x)(2I - Q) \\ &\quad - \left(\frac{1}{4\varepsilon} + \gamma\right)(F^{\rm T}(x)F(x) + \delta) \\ &\geq a(1 + \mu x^{\rm T}x)(2I - Q) \\ &\quad - \left(\frac{1}{4\varepsilon} + \gamma\right)(\bar{\gamma}^2x^{\rm T}x + \delta) \\ &\geq 0. \end{aligned} \tag{16}$$

Hence the Lyapunov difference given by (13) yields

$$\begin{aligned} \Delta V(x(k), \tilde{K}(k)) &\leq -\frac{\ell^{\rm T}(x(k))\ell(x(k))}{1 + V_{\rm s}(x(k))} \\ &\leq 0, \end{aligned} \tag{17}$$

which proves that the solution $(x(k), K(k)) \equiv (0, \hat{K}_{\rm g})$ to (8) and (10) is Lyapunov stable. Furthermore, it follows from (the discrete-time version of) Theorem 4.4 of [3] that $\ell(x(k)) \to 0$ as $t \to \infty$. Finally, if $\ell^{\rm T}(x)\ell(x) > 0$, $x \neq 0$, then $x(k) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

Remark 2.1. Theorem 2.1 is also valid for *time-varying* uncertain systems $\mathcal{G}_t$ of the form

[redacted]

$$x(k+1) = f(k, x(k)) + G(k, x(k))u(k) + J(k, x(k))w(k), \quad x(0) = x_0, \; k \in \mathcal{N}, \tag{18}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(k, 0) = 0$, $k \in \mathcal{N}$,

$\bar{\gamma}^2 x^{\mathrm{T}} x$, $k \in \mathcal{N}$, $x \in \mathbb{R}^n$, in place of (2), $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ by $\hat{G} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and requiring $G(k,x)\hat{G}(k,x)\Psi = J(k,x)$ in place of $G(x)\hat{G}(x)\Psi = J(x)$, it follows by using identical arguments as in the proof of Theorem 2.1 that the adaptive feedback control law

$$u(k) = \hat{G}(k, x(k))K(k)\tilde{x}(k), \tag{19}$$

where $\tilde{x}(k) \triangleq [F^{\mathrm{T}}(k, x(k)),\ w^{\mathrm{T}}(k)]^{\mathrm{T}}$, with update law

$$\begin{aligned} K(k+1) &= K(k) - Q\hat{G}^{-1}(k, x(k))G^{\dagger}(k, x(k)) \\ &\quad \cdot [x(k+1) - f_{\mathrm{c}}(x(k))]\tilde{x}^{\dagger}(k), \end{aligned} \tag{20}$$

where $f_{\mathrm{c}}(x) = f(k,x) + G(k,x)\hat{G}(k,x)K_{\mathrm{g}}F(k,x)$, guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\mathrm{g}},\ -\Psi])$ of the closed-loop system (18)–(20) is Lyapunov stable and $x(k) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Remark 2.2. It follows from Remark 2.1 that Theorem 2.1 can also be used to construct adaptive tracking controllers for nonlinear uncertain systems. Specifically, let $r_{\mathrm{d}}(k) \in \mathbb{R}^n$, $t \geq 0$, denote a command input and define the error state $e(k) \triangleq x(k) - r_{\mathrm{d}}(k)$. In this case, the error dynamics are given by

$$\begin{aligned} e(k+1) &= f_{\mathrm{t}}(k, e(k)) + G(k, e(k))u(k) \\ &\quad + J_{\mathrm{t}}(k, e(k))w_{\mathrm{t}}(k),\ e(0) = e_0,\ k \in \mathcal{N}, \end{aligned} \tag{21}$$

where $f_{\mathrm{t}}(k, e(k)) = f(e(k)+r_{\mathrm{d}}(k)) - n(k)$, with $f(r_{\mathrm{d}}(k)) = n(k)$, and $J_{\mathrm{t}}(k, e(k))w_{\mathrm{t}}(k) = n(k) - r_{\mathrm{d}}(k+1) + J(k, e(k))w(k)$. Now, the adaptive tracking control law (19) and (20), with $x(k)$ replaced by $e(k)$, guarantees that $e(k) \to 0$ as $t \to \infty$ for all $e_0 \in \mathbb{R}^n$.

It is important to note that the adaptive control law (7) and (8) does *not* require explicit knowledge of the gain matrix K_{g}, the disturbance matching matrix Ψ, the disturbance weighting matrix function $J(x)$, and the positive constants γ, $\bar{\gamma}$, ε and μ; even though Theorem 2.1 requires the existence of K_{g}, $F(x)$, $\hat{G}(x)$, and Ψ such that the zero solution $x(k) \equiv 0$ to (3) is globally asymptotically stable and the matching condition $G(x)\hat{G}(x)\Psi = J(x)$ holds. Furthermore, if (1) is in normal form with asymptotically stable internal dynamics [4] and if $f^{\mathrm{T}}(x)f(x) \leq \hat{\gamma}^2 x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\hat{\gamma} > 0$, then we can always construct a function $F : \mathbb{R}^n \to \mathbb{R}^s$ such that the zero solution $x(k) \equiv 0$ to (3) is globally asymptotically stable and (2) holds *without* requiring knowledge of the system dynamics. (For simplicity of exposition in the ensuing discussion we assume that $J(x) = D$, where $D \in \mathbb{R}^{n\times d}$ is a disturbance weighting matrix with unknown entries.) To see this assume that the nonlinear uncertain system $\mathcal{G}$ is generated by the difference model

$$\begin{aligned} z_i(k+\tau_i) &= f_{\mathrm{u}i}(z(k)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(z(k))u_j(k) \\ &\quad + \sum_{l=1}^{d} \hat{D}_{(i,l)}w_l(k), \quad z(0) = z_0, \quad k \in \mathcal{N}, \\ &\qquad\qquad i = 1, \cdots, m, \end{aligned} \tag{22}$$

where τ_i denotes the time delay (or relative degree) with respect to the output z_i, $f_{\mathrm{u}i}(z(k)) = f_{\mathrm{u}i}(z_1(k), \cdots, z_1(k+\tau_1-1), \cdots, z_m(k), \cdots, z_m(k+\tau_m-1))$, $G_{\mathrm{s}(i,j)}(z(k)) = G_{\mathrm{s}(i,j)}(z_1(k), \cdots, z_1(k+\tau_1-1), \cdots, z_m(k), \cdots, z_m(k+\tau_m-1))$, $\hat{D}_{(i,l)} \in \mathbb{R}$, $i = 1, \cdots, m$, $l = 1, \cdots, d$, and $w_l(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $l = 1, \cdots, d$. Here, we assume that the square matrix function $G_{\mathrm{s}}(z)$ composed of the entries $G_{\mathrm{s}(i,j)}(z)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(z) \neq 0$, $z \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \cdots + r_m$. Furthermore, since (22) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$.

Next, define $x_i(k) \triangleq [\, z_i(k), \cdots, z_i(k+\tau_i-2)]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1}(k) \triangleq [\, z_1(k+\tau_1-1), \cdots, z_m(k+\tau_m-1)]^{\mathrm{T}}$, and $x(k) \triangleq [x_1^{\mathrm{T}}(k), \cdots, x_{m+1}^{\mathrm{T}}(k)]^{\mathrm{T}}$, so that (22) can be described by (1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \tag{23}$$

$$J(x) = D = \begin{bmatrix} 0_{(n-m)\times d} \\ \hat{D} \end{bmatrix}, \tag{24}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [5], $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function and satisfies $f_{\mathrm{u}}^{\mathrm{T}}(x)f_{\mathrm{u}}(x) \leq \gamma_{\mathrm{u}}^2 x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, where $\gamma_{\mathrm{u}} > 0$, $G_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and $\hat{D} \in \mathbb{R}^{m\times d}$. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}^{\mathrm{T}}(x)f_{\mathrm{n}}(x) \leq \gamma_{\mathrm{n}}^2 x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\gamma_{\mathrm{n}} > 0$, and $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters.

Next, to apply Theorem 2.1 to the uncertain system (1) with $f(x)$, $G(x)$, and D given by (23) and (24), let $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$K_{\mathrm{g}} = [\Theta_{\mathrm{n}} - \Theta,\ \Phi_{\mathrm{n}}], \tag{25}$$

where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_{\mathrm{n}}(x) \\ \hat{f}_{\mathrm{n}}(x) \end{bmatrix}, \tag{26}$$

where $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ and satisfies $\hat{f}_{\mathrm{n}}^{\mathrm{T}}(x)\hat{f}_{\mathrm{n}}(x) \leq \hat{\gamma}_{\mathrm{u}}^2 x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\hat{\gamma}_{\mathrm{u}} > 0$, is an arbitrary function. In this case, it follows that, with $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$,

$$\begin{aligned} f_{\mathrm{c}}(x) &= f(x) + G(x)\hat{G}(x)K_{\mathrm{g}}F(x) \\ &= \tilde{A}x + \tilde{f}_{\mathrm{u}}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix} G_{\mathrm{s}}^{-1}(x) \\ &\quad \cdot [\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) - \Theta f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}}\hat{f}_{\mathrm{n}}(x)\ ] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}}\hat{f}_{\mathrm{n}}(x) \end{bmatrix}. \end{aligned} \tag{27}$$

Note that, with $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$, Ψ in Theorem 2.1 can be taken as $\Psi = \hat{D}$ so that $G(x)\hat{G}(x)\Psi = J(x) = D$ is satisfied, and (2) is satisfied with $\bar{\gamma}^2 \geq \gamma_{\mathrm{n}}^2 + \hat{\gamma}_{\mathrm{n}}^2$.

Now, since $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct K_{g} and $F(x)$ without knowledge of $f(x)$ such that the zero solution $x(t) \equiv 0$ to (3) can be made globally asymptotically stable. In particular, choosing $\Theta_{\rm n} f_{\rm n}(x) + \Phi_{\rm n} \hat{f}_{\rm n}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that (27) has the form $f_{\rm c}(x) = A_{\rm c}x$, where $A_{\rm c} = \left[A_0^{\rm T}, \hat{A}^{\rm T}\right]^{\rm T}$ is in multivariable controllable canonical form. Hence, in the case where $G(x)\hat{G}(x) \equiv B$ is a constant matrix, by choosing $\hat{A}$ such that $A_{\rm c}$ is asymptotically stable it follows that for sufficiently small ε there exists a positive-definite matrix P satisfying the following Riccati-type inequality

$$0 \geq A_{\rm c}^{\rm T} P A_{\rm c} - P + R + 4\varepsilon A_{\rm c}^{\rm T} P B B^{\rm T} P A_{\rm c}, \qquad (28)$$

where R is positive definite. In this case, with Lyapunov function $V_{\rm s}(x) = x^{\rm T}Px$, (4)–(6) are satisfied with $P_{1\tilde{u}}(x) = 2x^{\rm T}A_{\rm c}^{\rm T}PB$, $P_{2\tilde{u}}(x) = B^{\rm T}PB$, and $\mu \leq \lambda_{\min}(P)$, and hence the adaptive feedback controller (7) with update law (8) guarantees global asymptotic stability of the nonlinear uncertain discrete-time dynamical system (1) where $f(x)$, $G(x)$, and $J(x)$ are given by (23) and (24). As mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F(x)$ to produce a linear $f_{\rm c}(x)$. However, when the system is in normal form, a feedback linearizing function $F(x)$ assures the existence of $V_{\rm s}(x)$ that satisfies the conditions (4) and (5). In particular, choosing $\Theta_{\rm n} = \Phi_{\rm n} = 0$, it follows that

$$G^{\dagger}(x)f_{\rm c}(x) = \left[\ 0_{m\times(n-m)},\quad G_{\rm s}^{-1}(x)\ \right]\left[\begin{array}{c} A_0 \\ 0_{m\times n}\end{array}\right]x = 0,$$

and hence the update law (8) can be simplified as

$$K(k+1) = K(k) - Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))x(k+1)\tilde{x}^{\dagger}(k). \qquad (29)$$

Finally, note that Theorem 2.1 is not restricted to systems with sector-bounded nonlinearities so long as the regressor function $F(x)$ satisfies (2) and we can construct a known function $f_{\rm c}(x)$ such that the zero solution $x(k) \equiv 0$ to (3) is globally asymptotically stable.

Next, we consider the case where $f(x)$ and $G(x)$ are uncertain. Specifically, we assume that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$ and $\sigma_{\max}(B_{\rm u}) < 2$, is an unknown symmetric sign definite matrix but the sign definiteness of $B_{\rm u}$ is known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\rm T}$ for $B_{\rm u} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\rm T}$ for $B_{\rm u} < 0$.

Corollary 2.1. Consider the nonlinear system $\mathcal{G}$ given by (1) with $f(x)$, $G(x)$, and $J(x)$ given by (23) and (24), and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $B_{\rm s}$ is an unknown symmetric matrix and the sign definiteness of $B_{\rm u}$ is known. Assume there exists a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$ and a function $F : \mathbb{R}^n \to \mathbb{R}^s$ such that the zero solution $x(k) \equiv 0$ to (3) is globally asymptotically stable and (2) holds. Furthermore, assume there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $P_{1\tilde{u}} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $\ell : \mathbb{R}^n \to \mathbb{R}^t$, and a nonnegative-definite matrix function $P_{2\tilde{u}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ such that $P_{2\tilde{u}}(x) \leq \gamma I_m$, $x \in \mathbb{R}^n$, $\gamma > 0$, $V_{\rm s}(\cdot)$ is continuous, positive definite, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$ and $\tilde{u} \in \mathbb{R}^m$, (4)–(6) hold. Finally, let $\tilde{x}(k) \triangleq [F^{\rm T}(x(k)), w^{\rm T}(k)]^{\rm T}$. Then the adaptive feedback control law

$$u(k) = G_{\rm n}^{-1}(x(k))K(k)\tilde{x}(k), \qquad (30)$$

where $K(k) \in \mathbb{R}^{m\times(s+d)}$, $k \in \mathcal{N}$, and $\tilde{x}(k) \triangleq [F^{\rm T}(x(k)), w^{\rm T}(k)]^{\rm T}$, with update law

$$K(k+1) = K(k) - B_0^{\rm T}[x(k+1) - f_{\rm c}(x(k))]\tilde{x}^{\dagger}(k), \qquad (31)$$

guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\rm g}, -\Psi])$, where $\Psi \in \mathbb{R}^{m\times d}$, of the closed-loop system given by (1), (30), and (31) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. The result is a direct consequence of Theorem 2.1. First, let $\hat{G}(x) = G_{\rm n}^{-1}(x)$ and $\Psi = B_{\rm u}^{-1}\hat{D}$ so that $G(x)\hat{G}(x) = [0_{m\times(n-m)}, B_{\rm u}]^{\rm T}$ and $G(x)\hat{G}(x)\Psi = D$, and let $K_{\rm g} = B_{\rm u}^{-1}[\Theta_{\rm n} - \Theta, \Phi_{\rm n}]$. Next, since Q in (8) is an arbitrary positive-definite matrix with $\lambda_{\max}(Q) < 2$, it can be replaced by $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where U is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $|B_{\rm u}|G_{\rm n}(x)G^{\dagger}(x) = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm s} < 0$. □

3. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed discrete-time adaptive control framework for adaptive stabilization, disturbance rejection, and command following.

Example 3.1. Consider the linear uncertain system given by

$$z(k+2) + a_1 z(k+1) + a_0 z(k) = bu(k) + \hat{d}\sin 7k,$$
$$z(0) = z_0,\ z(1) = z_1,\ k \in \mathcal{N}, \qquad (32)$$

where $z(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $u(k) \in \mathbb{R}$, $k \in \mathcal{N}$, and $a_0, a_1, b, \hat{d} \in \mathbb{R}$ are unknown constants. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (32) can be written in state space form (1) with $x = [x_1, x_2]^{\rm T}$, $f(x) = [x_2, -a_0x_1 - a_1x_2]^{\rm T}$, $G(x) = [0, b]^{\rm T}$, $J(x) = [0, \hat{d}]^{\rm T}$, and $w(k) = \sin 7k$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1x_1 + \theta_2x_2]^{\rm T}$, where θ_1 and θ_2 are unknown constants. Furthermore, we assume that $\operatorname{sign} b$ is known and $|b| < 2$. Next, let $G_{\rm n}(x) = 1$, $F(x) = x$, and $K_{\rm g} = \frac{1}{b}\left[\theta_{\rm n_1} - \theta_1, \theta_{\rm n_2} - \theta_2\right]$, where $\theta_{\rm n_1}, \theta_{\rm n_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\rm c}(x) &= f(x) + \left[\begin{array}{c} 0 \\ b \end{array}\right]\frac{1}{b}\left[\ \theta_{\rm n_1} - \theta_1,\ \theta_{\rm n_2} - \theta_2\ \right]F(x) \\ &= \left[\begin{array}{cc} 0 & 1 \\ \theta_{\rm n_1} & \theta_{\rm n_2}\end{array}\right]x. \end{aligned}$$

Now, with the proper choice of $\theta_{\rm n_1}$ and $\theta_{\rm n_2}$, it follows from Corollary 2.1 that the adaptive feedback controller (30) guarantees that $x(k) \to 0$ as $k \to \infty$. With $\theta_1 = -1$, $\theta_2 = 0.25$, $b = 0.4$, $\hat{d} = 10$, $\theta_{\rm n_1} = -0.02$, $\theta_{\rm n_2} = 0.3$, and initial conditions $x(0) = [-1, 3]^{\rm T}$ and $K(0) = [0, 0, 0]$, Figure 3.1 shows that the phase portrait of the controlled and

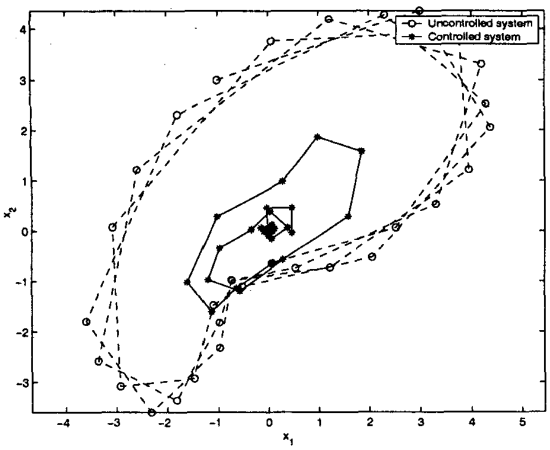

Figure 3.1: Phase portrait of controlled and uncontrolled system

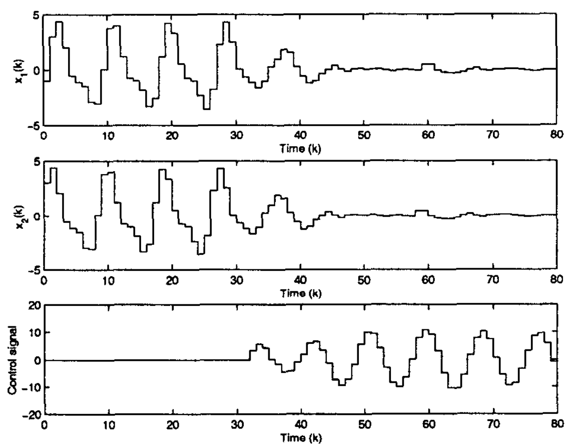

Figure 3.2: State trajectories and control signal versus time

uncontrolled system. Note that the adaptive controller is switched on at $k = 30$. Figure 3.2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 3.3 shows the adaptive gain history versus time.

Example 3.2. Consider the two-degree of freedom uncertain nonlinear system given by

$$M_s z(k+2) + C_s z(k+1) + K_s z(k) = u(k), \quad z(0) = z_{00},\ z(1) = z_{10},\ k \in \mathcal{N}, \quad (33)$$

where $z(k) \in \mathbb{R}^2$, $u(k) \in \mathbb{R}^2$, $k \in \mathcal{N}$, $M_s, C_s, K_s \in \mathbb{R}^{2\times 2}$. Here we assume that $M_s = M_s^{\mathrm{T}} > 0$ and $\sigma_{\max}(M_s^{-1}) < 2$. Let $r_d(k)$ be a desired command signal and define the error state $\tilde{e}(k) \triangleq z(k) - r_d(k)$ so that the error dynamics are given by

$$M_s\tilde{e}(k+2) + C_s\tilde{e}(k+1) + K_s\tilde{e}(k) = u(k) - M_s r_d(k+2) - C_s r_d(k+1) - K_s r_d(k),$$

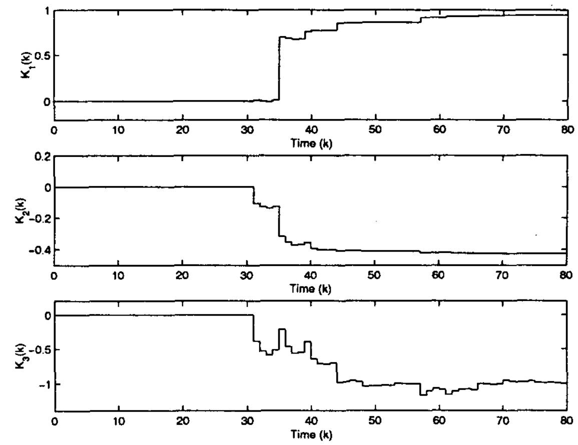

Figure 3.3: Adaptive gain history versus time

$$\tilde{e}(0) = \tilde{e}_0,\ \tilde{e}(1) = \tilde{e}_1,\ k \in \mathcal{N}. \quad (34)$$

Note that with $e_1(k) = \tilde{e}(k)$ and $e_2(k) = \tilde{e}(k+1)$, (34) can be written in state space form (21) with $e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}]^{\mathrm{T}}$, $f_t(k, e) = [e_2^{\mathrm{T}}, -(M_s^{-1}K_s e_1 + M_s^{-1}C_s e_2)^{\mathrm{T}}]^{\mathrm{T}}$, $G(k, e) = [0_{2\times 2}, M_s^{-1}]^{\mathrm{T}}$, $J_t(k, e) = [0_{6\times 2}, \hat{D}_t^{\mathrm{T}}]^{\mathrm{T}}$, where $\hat{D}_t = [-I_2, -M_s^{-1}C_s, -M_s^{-1}K_s]$, and $w_t(k) = [r_d^{\mathrm{T}}(k+2), r_d^{\mathrm{T}}(k+1), r_d^{\mathrm{T}}(k)]^{\mathrm{T}}$. Note that M_s^{-1} is symmetric and positive definite but unknown. Next, let $K_g = M_s[\Theta_{n_1} + M_s^{-1}K_s, \Theta_{n_2} + M_s^{-1}C_s]$, where $\Theta_{n_1} \in \mathbb{R}^{2\times 2}$, $\Theta_{n_2} \in \mathbb{R}^{2\times 2}$ are arbitrary matrices, so that

$$f_c(e) = \begin{bmatrix} 0_2 & I_2 \\ \Theta_{n_1} & \Theta_{n_2} \end{bmatrix} e.$$

Now, with the proper choice of Θ_{n_1} and Θ_{n_2}, it follows from Corollary 2.1 and Remark 2.2 that the adaptive feedback controller (30) guarantees that $e(k) \to 0$ as $t \to \infty$. With

$$M_s = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad C_s = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad K_s = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$r_d(k) = [\sin 0.5k, 0.5]^{\mathrm{T}}$, $\Theta_{n_1} = \Theta_{n_2} = 0_2$, and initial conditions $x(0) = [3, -4, -2, 1]^{\mathrm{T}}$ and $K(0) = 0_{2\times 10}$, Figure 3.4 shows the actual positions and the reference signals versus time and the control signals versus time. Note that the adaptive controller is switched on at $k = 40$.

4. Conclusion

A discrete-time direct adaptive nonlinear control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain systems with exogenous bounded disturbances was developed. Using Lyapunov methods the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable zero dynamics, the nonlinear adaptive controllers were constructed without knowledge of the system dynamics. Finally, two illustrative numerical examples were presented to show the utility of the proposed adaptive stabilization and tracking scheme.

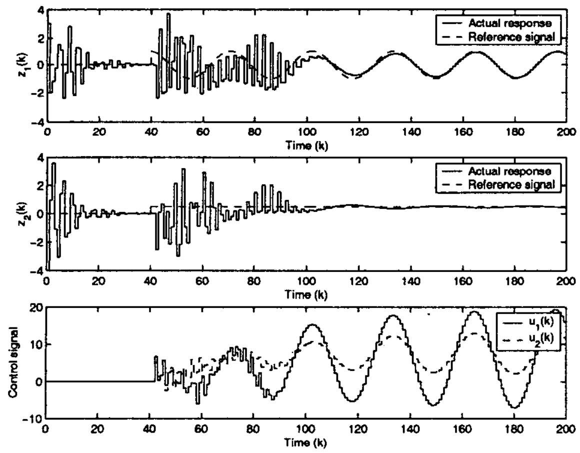

Figure 3.4: Positions and control signals versus time

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